MMP 2
Solutions 7

FS 19
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Exercise 1

a)

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} \qquad n \in \mathbb{N}$$

let $$f(z) = \frac{1}{(1+z^2)^n}$$

then

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} = \int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum \text{residules of upper half-plane} \tag{1}$$

$$f(z) = \frac{1}{(z-i)^n(z+i)^n} \Rightarrow \text{pole of order } n \text{ at } i \text{ in upper half-plane}$$

$$\Rightarrow Res(f,i) = \lim_{z\to i} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}[(z-i)^n f(z)]$$

$$= \frac{1}{(n-1)!} \lim_{z\to i} \frac{\partial^{(n-1)}}{\partial z^{(n-1)}} \frac{1}{(z+i)^n}$$

$$= \frac{1}{(n-1)!} \lim_{z\to i} -(n) \cdot -(n+1) \cdot -(n+2) \cdot ... \cdot -(n+n-2)(z+i)^{-n-n+1}$$

$$= \frac{(-1)^{n-1}}{(n-1)!} \frac{(2n-2)!}{(n-1)!} (2i)^{-2n+1} \tag{2}$$

Sub (2) into (1):

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} = 2\pi i \frac{(-1)^{n-1}}{((n-1)!)^2}(2n-2)!(2i)^{-2n+1}$$

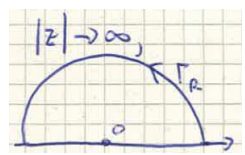
$$\boxed{= \pi \frac{(2n-2)!}{((n-1)!)^2} 2^{-2n+2}}$$

b)

$$\int_{0}^{\infty} \frac{\cos(x)}{x^2+a^2}dx \underset{\substack{\uparrow \\ \text{even function}}}{=} \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}dx = \frac{1}{2}Re\left[\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x^2+a^2}dx\right]$$

We go in the complex plane, $\frac{e^{ix}}{x^2+a^2} \longrightarrow \frac{e^{iz}}{z^2+a^2}$

and integrate over the upper half-plane.
The function vanishes with order 2 for $|z| \to \infty$ so Γ_R does not contrlate to the integral



With the residue theorem we find:

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x^2+a^2}dx = 2\pi i \underset{z=ai}{Res}\left(\frac{e^{iz}}{(z-ai)(z+ai)}\right) = \frac{\pi}{a}e^{-a}$$

Therefore, $$\int_{0}^{\infty} \frac{\cos(x)}{x^2+a^2}dx = \frac{\pi}{2a}e^{-a}$$

Exercise 2

a)

$$(z^2+z)y''(z)+y(z)=0 \tag{1}$$

$$\Rightarrow y''(z)+\frac{1}{z(1+z)}y(z)=0$$

$$\Rightarrow P(z)=0 \qquad Q(z)=\frac{1}{z(z+1)}$$

Has singularities $z=-1,0,\infty$

$z=-1:$

$$\begin{aligned} Q(z) &= -\frac{1}{z+1}-\frac{1}{1-(z+1)} \\ &= -\frac{1}{z+1}-\sum_{n=0}^{\infty}(z+1)^n \qquad (|z+1|<1) \\ &= -\sum_{n=-1}^{\infty}(z+1)^n \end{aligned}$$

expansion coefficients are 0 for $n<-1$

$\Rightarrow$ <u>simple pole for $z=-1$</u>

$z=0:$

$$\begin{aligned} Q(z) &= \frac{1}{z}-\frac{1}{z+1} \\ &= \frac{1}{z}-\sum_{n=0}^{\infty}(-z)^n \qquad (|z|<1) \\ &= \sum_{n=-1}^{\infty}(-1)^{n+1}z^n \end{aligned}$$

expansion coefficients are 0 for $n<-1$

$\Rightarrow$ <u>simple pole for $z=0$</u>

$z \to \infty$:

To determine what type of singularity we have at infinity we substitute $z = \frac{1}{\xi}$. We transform the derivative operators to express the differential equation in terms of ξ and write it in standard form:

$$\frac{d}{dz} = \frac{d\xi}{dz}\frac{d}{d\xi} = -\frac{1}{z^2}\frac{d}{d\xi} = -\xi^2\frac{d}{d\xi}, \qquad \frac{d^2}{dz^2} = -\xi^2\frac{d}{d\xi}\left(-\xi^2\frac{d}{d\xi}\right) = \xi^4\frac{d^2}{d\xi^2} + 2\xi^3\frac{d}{d\xi}$$

$$y''(z) + P(z)y'(z) + Q(z)y(z) = \left[\xi^4\frac{d^2}{d\xi^2} + 2\xi^3\frac{d}{d\xi} - P\left(\frac{1}{\xi}\right)\xi^2\frac{d}{d\xi} + Q\left(\frac{1}{\xi}\right)\right]y\left(\frac{1}{\xi}\right) = 0$$

$$\text{standard form:} \quad \left[\frac{d^2}{d\xi^2} + \underbrace{\frac{2\xi - P\left(\frac{1}{\xi}\right)}{\xi^2}}_{\tilde{P}(\xi)}\frac{d}{d\xi} + \underbrace{\frac{Q\left(\frac{1}{\xi}\right)}{\xi^4}}_{\tilde{Q}(\xi)}\right]y\left(\frac{1}{\xi}\right) = 0$$

We can now examine the poles of $\tilde{P}(\xi)$ and $\tilde{Q}(\xi)$ in $\xi = 0$ since this corresponds to $z \to \infty$:

$$\lim_{\xi\to 0} z(\xi) = \lim_{\xi\to 0}\frac{1}{\xi} \to \infty$$

$$\tilde{P}(\xi) = \frac{2\xi - P(\frac{1}{\xi})}{\xi^2} = \frac{2}{\xi} \overset{\xi\to 0}{\sim} \frac{1}{\xi} \qquad \tilde{Q}(\xi) = \frac{Q(\frac{1}{\xi})}{\xi^4} = \frac{\xi}{\xi^4(1+\frac{1}{\xi})} = \frac{1}{\xi^2(\xi+1)} \overset{\xi\to 0}{\sim} \frac{1}{\xi^2}$$

$$\Rightarrow \quad \underline{\text{non essential singularity at } z \to \infty}$$

b) Mobius transformation:

$$f(z) = \frac{az+b}{cz+d} \qquad ad \neq bc$$

let $\quad a = -1, \quad b = 0, \quad c = 0, \quad d = 1$

$$\Rightarrow f(z) = -z$$

define
$$\begin{aligned} \tilde{y}(z) &= y(f(z)) \\ &= y(-z) \\ \Rightarrow \tilde{y}''(z) &= y''(-z) \end{aligned}$$

$\Rightarrow$ (1) becomes:

$$\tilde{y}''(z) + \frac{1}{z(z-1)}\tilde{y}(z) = 0 \tag{4}$$

Same form as the hypergeometric function:

$$z(1-z)\tilde{y}'' - [\gamma - (\alpha+\beta+1)z]\,\tilde{y}' - \alpha\beta\tilde{y} = 0$$

OR:
$$\tilde{y}'' - \frac{[\gamma - (\alpha+\beta+1)z]}{z(1-z)}\tilde{y}' + \frac{\alpha\beta}{z(z-1)}\tilde{y} = 0$$

Where here:

$$\begin{aligned} \alpha\beta &= 1 \\ \gamma &= 0 \\ \alpha+\beta+1 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \alpha &= \frac{-1+i\sqrt{3}}{2} \\ \beta &= \frac{-1-i\sqrt{3}}{2} \\ \gamma &= 0 \end{aligned} \tag{2}$$

Two solutions are, ${}_2F_1(\alpha\ ,\ \beta\ ;\ \gamma\ ;\ z)$
and $Z^{1-\gamma}\ {}_2F_1(1+\alpha-c\ ,\ 1+\beta-\gamma\ ;\ 2-\gamma\ ;\ z)$ (3)
around $z=0$, where
${}_2F_1(a\ ,\ b;\ c\ ;\ z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}$
where Γ is the Gamma function and $c \neq 0$
Since $\gamma = 0$, Solution (2) is invalid.

(3) still works

$$\begin{aligned} \text{Let } \tilde{y}_1 &= Z\,{}_2F_1(1+\alpha\ ,\ 1+\beta\ ;\ 2\ ;\ z) \\ &= ZF \\ \Rightarrow \tilde{y}_1' &= 2F' + ZF'' \text{ and } \tilde{y}_1' = F + ZF' \end{aligned}$$

by (4):

$$2F' + ZF'' + \frac{F}{z-1} = 0 \tag{5}$$

Let's try: $\quad \tilde{y}_2(z) = (ZF)g(z) \qquad g$ some function

$$\Rightarrow \quad \tilde{y}_2'' = (2F' + ZF'')g + 2(F+ZF')g' + (ZF)g''$$

By (4):

$$
\begin{aligned}
0 &= (2F' + ZF'')g + 2(F + ZF')g + ZFg'' + \frac{Fg}{Z-1} \\
&= 2(F + ZF')g' + ZFg'' \qquad \text{by (5)} \\
\Rightarrow \int_0^{g'} \frac{1}{g'} dg' &= -2 \int_0^{Z} \frac{1}{Z} + \frac{F'}{F} dZ \\
\ln g' &= -2(\ln z + \ln F) \\
g' &= (ZF)^{-2} \\
\Rightarrow g &= \int_0^{Z} (ZF)^{-2} dZ
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
y(z) &= -AZF + BZF \int_0^{Z} (ZF)^{-2} dZ \\
&A, B \text{ constants,} \\
&F =_2 F_1(\frac{1}{2} + \frac{i\sqrt{3}}{2}\ ,\ \frac{1}{2} - \frac{i\sqrt{3}}{2}\ ,\ 2\ ;\ -z)
\end{aligned}
}
$$

Exercise 3

a)

$$4z^2 y''(z) + y(z) = 0 \tag{1}$$

i) (1) becomes:

$$
\begin{aligned}
y''(z) + \frac{1}{4z^2} y(z) &= 0 \\
\Rightarrow P(z) = 0, \quad Q(z) &= \frac{1}{4z^2} \\
\text{singularities at } &= 0, \infty
\end{aligned}
$$

$$
\begin{aligned}
z = 0: \qquad Q(z) &= \frac{1}{4z^2} \\
&= \frac{1}{4} \sum_{n=-2}^{-2} z^n
\end{aligned}
$$

the coefficients of the expansion vanish for $n < -2$

$\Rightarrow$ <u>a pole of second order at $z = 0$, so not essential</u>

$$z \to \infty: \qquad \tilde{P}(\xi) = \frac{2\xi - P(\frac{1}{\xi})}{\xi^2} = \frac{2}{\xi} \overset{\xi \to 0}{\sim} \frac{1}{\xi} \qquad \tilde{Q}(\xi) = \frac{Q(\frac{1}{\xi})}{\xi^4} = \frac{1}{4\xi^2} \overset{\xi \to 0}{\sim} \frac{1}{\xi^2}$$

$\underline{z \to \infty \text{ is a non-essential singular point}}$

ii) Ansatz around $z = 0$:

$$\underline{y(z) = \sum_{n=0}^{\infty} a_n z^{\varrho+n}, \; a_0 \neq 0} \tag{2}$$

Radius of convergence: $\boxed{\infty}$

iii)

$$y' = \sum_{n=0}^{\infty} a_n(\varrho + n) z^{\varrho+n-1} \qquad \text{from (2)}$$

$$y'' = \sum_{n=0}^{\infty} a_n(\varrho + n)(\varrho + n - 1) z^{\varrho+n-2} \tag{3}$$

sub (2) + (3) into (1):

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} a_n(\varrho + n)(\varrho + n - 1) z^{\varrho+n-2} + \frac{1}{4z^2} a_n z^{\varrho+n} \\ &= \sum_{n=0}^{\infty} a_n((\varrho + n)(\varrho + n - 1) + \frac{1}{4}) z^{\varrho+n-2} \end{aligned}$$

$$\text{for } n = 0: \quad \varrho(\varrho - 1) + \frac{1}{4} = 0$$

$$\Rightarrow \underline{\varrho = \frac{1}{2}}$$

iv) For $n > 0$, $\varrho = \frac{1}{2}$:

$$\begin{aligned} &(\frac{1}{2} + n)(\frac{1}{2} + n - 1) + \frac{1}{4} = 0 \\ &\frac{1}{4} + \frac{1}{2}n - \frac{1}{2} + \frac{1}{2}n + n^2 - n + \frac{1}{4} = 0 \\ \Rightarrow \quad &a_n = 0 \quad \text{for } n > 0 \\ &\underline{y_1 = a_0\sqrt{z}} \end{aligned}$$

v) Ansatz: $\qquad y_2(z) = y_1(z) \ln(z) \qquad (4)$

$$
\begin{aligned}
y_2' &= y_1' \ln(z) + \frac{y_1}{z} \\
&= z^{-\frac{1}{2}}(\frac{1}{2}\ln(z) + 1) \\
y_2'' &= -\frac{1}{2}z^{-\frac{3}{2}}(\frac{1}{2}\ln(z) + 1) + z^{-\frac{1}{2}}(\frac{1}{2} \cdot \frac{1}{z}) \\
&= -\frac{1}{4}z^{-\frac{3}{2}}\ln(z) \qquad (5)
\end{aligned}
$$

sub (4) + (5) into LHS for (1):

$$
\begin{aligned}
-\frac{1}{4}z^{-\frac{3}{2}}\ln(z) + \frac{1}{4z^2}(z^{\frac{1}{2}}\ln(z)) \\
&= -(\frac{z^{-\frac{3}{2}}}{4})\ln(z) + (\frac{z^{-\frac{3}{2}}}{4})\ln(z) \\
&= 0 \\
\Rightarrow \quad y(z) &= C\sqrt{z} + D\sqrt{z}\ln(z) \\
&C, D \text{ constants}
\end{aligned}
$$

b) i) $zy''(z) + 2y'(z) - zy(z) = 0$ (1)

$$\Rightarrow\ y'' + \frac{2}{z}y' - y = 0 \quad \Rightarrow\ P(z) = \frac{2}{z}, \quad Q(z) = -1 \quad \Rightarrow \text{singular points: } z = 0,\ \infty$$

$z = 0:$ $\quad P(z) = \frac{2}{z}$, Expansion is 0 for exponents < -1 $\Rightarrow$ <u>simple pole at $z = 0$</u>

$z \to \infty:$ $\quad \tilde{P}(\xi) = \frac{2\xi - P(\frac{1}{\xi})}{\xi^2} = 0 \overset{\xi\to 0}{\sim} 0, \qquad \tilde{Q}(\xi) = \frac{Q(\frac{1}{\xi})}{\xi^4} = -\frac{1}{\xi^4} \overset{\xi\to 0}{\sim} \frac{1}{\xi^4}$

<u>essential singularity at $z \to \infty$</u>

ii) Ansatz around $z = 0$:

$$\underline{y(z) = \sum_{n=0}^{\infty} a_n z^{\varrho+n}} \qquad a_0 \neq 0 \ \ (\text{And } a_n = 0 \text{ for } n > 0) \tag{2}$$

Radius of convergence: $\boxed{\infty}$

iii)

$$\left.\begin{aligned}
y' &= \sum_{n=0}^{\infty} a_n(\varrho+n)z^{\varrho+n+1} \\
y'' &= \sum_{n=0}^{\infty} a_n(\varrho+n)(\varrho+n-1)z^{\varrho+n-2}
\end{aligned}\right\} \tag{3}$$

sub (2) + (3) into (1):

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n(\varrho+n)(\varrho+n-1)z^{\varrho+n-2} + \frac{2}{z}a_n(\varrho+n)z^{\varrho+n+1} - a_{n-2}z^{\varrho+n-2} \\
&= \sum_{n=0}^{\infty} z^{\varrho+n-2}\left[a_n(\varrho+n)(\varrho+n-1) + 2a_n(\varrho+n) - a_{n-2}\right]
\end{aligned}
$$

For $n = 0$: $\quad \varrho(\varrho - 1) + 2(\varrho) = 0 \quad \Rightarrow \boxed{\varrho = 0,\ -1}$

iv) For $n > 0$, $\varrho = -1$: From (5):

$$\begin{aligned} 0 &= a_n(n-1)(n-2) + 2a_n(n-1) - a_{n-2} \\ &= a_n\left[(n-1)(n-2) + 2(n-1)\right] - a_{n-2} \\ \Rightarrow \quad a_n &= \frac{a_{n-2}}{(n-1)n} \end{aligned}$$

Even n, a_0 arbitrary

$$\begin{aligned} a_n &= \frac{a_{n-4}}{n(n-1)(n-2)(n-3)} \\ &= \frac{a_{n-6}}{n(n-1)(n-2)(n-3)(n-4)(n-5)} \\ &= \ldots = \frac{a_0}{n!} \\ \Rightarrow a_{2n} &= \frac{a_0}{(2n)!} \end{aligned}$$

Similarly for odd n, a, arbitrary

$$a_{2n+1} = \frac{a_1}{(2n+1)!}$$

$$\Longrightarrow \quad \boxed{\begin{gathered} y(z) = a_0 \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)!} \\ \textbf{OR} \\ y(z) = \frac{1}{z}\left[a_0 \cosh z + a_1 \sinh z\right] \end{gathered}}$$