General Relativity
Sheet 12

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Exercise 1 [Reissner-Nordström solution: charged black holes]

In this exercise you will derive a solution of Einstein's equations for charged bodies. We will do that by closely following the derivation of the Schwarzschild metric presented in the lecture. By staticity and spherical symmetry, this solution will have the general form (we use units where $G = c = 1$)

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{1}$$

as presented in the lecture. In contrast to the Schwarzschild case where we dealt with a vacuum, there is now a non-vanishing contribution from the stress-energy tensor of the electromagnetic field.

Consider the source-free ($j^\mu = 0$) Maxwell equations in a static and spherically symmetric spacetime.

a) Argue that the most general form of an electromagnetic field-strength tensor that shares the static and spherical symmetries of the spacetime has non-zero components (in the usual Schwarzschild coordinates) $F_{tr} = -F_{rt} = f(r)$ and $F_{\theta\phi} = -F_{\phi\theta} = g(r)\sin\theta$, where $f(r)$ and $g(r)$ are functions of the radial coordinate r.

Assuming there are no magnetic monopoles, show that $g(r) = 0$.

Hint: Use the symmetry conditions $L_K F_{\mu\nu} = 0$, where $K = R, S, T$ are the Killing vectors of S^2

$$R = \partial_\phi \tag{2}$$
$$S = \cos\phi \, \partial_\theta - \cot\theta \sin\phi \, \partial_\phi \tag{3}$$
$$T = -\sin\phi \, \partial_\theta - \cot\theta \cos\phi \, \partial_\phi. \tag{4}$$

b) The (traceless) stress-energy tensor of the electromagnetic field is given by

$$T^{\text{em}}_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\gamma}F^{\gamma}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\gamma\delta}F^{\gamma\delta}\right]. \tag{5}$$

Compute its components in terms of $A(r)$, $B(r)$ and $f(r)$ by using the given metric and your result from part a).

c) In the lecture you have computed the components of the Ricci tensor in a static and spherically symmetric spacetime. Plug them into the Einstein equations

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{6}$$

and use the energy momentum tensor you found in b). Try to use the same trick as in the lecture to show that $A(r)B(r) = c$. Moreover, show that

$$B(r) = c - \frac{2a}{r} - \frac{1}{r}\int r^2 f^2(r) \, \mathrm{d}r, \tag{7}$$

where a and c are constants.

d) Use the $\mu = t$ component of Maxwell's equations $\nabla_\nu F^{\mu\nu} = 0$ together with (7) to show that the Reissner-Nordström metric can be expressed as

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2, \qquad (8)$$

where Q is the total charge of the body. This result is very important: The electromagnetic field contains energy and is therefore also a source of a gravitational field. Through this mechanism, the charge of a black hole is able to change the trajectories even of particles that do not carry a charge.

e) Use the program you wrote for the last exercise sheet to calculate the two curvature invariants

- $R^{\mu\nu} R_{\mu\nu}$
- $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$.

What does this tell you about the singularities?

f) The surfaces on which the metric experiences a coordinate singularity correspond to event horizon surfaces. Calculate the possible position(s) of the event horizon(s) as a function of charge and mass. Discuss the different cases that are possible.