Exercise 1. *The soliton of $\lambda\phi^4$ theory in $1+1$ dimensions*

Consider the scalar field theory with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi), \qquad V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\,, \tag{1}$$

in one dimension ($g_{\mu\nu} = \mathrm{diag}(+1,-1)$) with $m^2 < 0$.

1. *Determine the constant solutions of the equations of motion and shift the potential in such a way that they have vanishing energy.*
2. *Find the static solutions of the equations of motion that interpolate between two constant solutions (these interpolating solutions are called solitons). Use the ansatz $\phi(x) = a\tanh(bx)$.*
3. *Calculate the energy of the static soliton (Hint: $\int_{-\infty}^{\infty} \mathrm{d}x \cosh^{-4}(x) = 4/3$).*
4. *Check that the current*
$$J^\mu = \epsilon^{\mu\nu}\partial_\nu \phi \tag{2}$$
is conserved. What are the possible values of $\int \mathrm{d}x J^0$ for a solution of the equations of motion?

Solution.

1. We study the equation of motions for the scalar field ϕ,
$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{\partial V}{\partial \phi} = 0\,. \tag{S.1}$$
The constant solution are corresponds to the extreme points of the potential,
$$\frac{\partial V}{\partial \phi} = 0\,. \tag{S.2}$$
For $V = -\frac{1}{2}|m|^2\phi^2 + \frac{\lambda}{4!}\phi^4$, we find
$$\frac{\partial V}{\partial \phi} = \frac{\lambda}{6}\phi\left(\phi^2 - \frac{6|m|^2}{\lambda}\right) = 0\,, \tag{S.3}$$
which admits solutions
$$\phi_0 = \{0, \pm v\}\,, \quad \text{with} \quad v = \frac{\sqrt{6}|m|}{\sqrt{\lambda}}\,. \tag{S.4}$$
$\phi_0 = \pm v$ correspond the two degenerate minima of the potential $V(\pm v) = -\frac{3|m|^4}{2\lambda}$, which are separated by the local maximum $V(0) = 0$. In order to reset to zero the energy of the non-trivial constant solutions, we perform a shift of the potential
$$\begin{aligned} V'(\phi) =& V(\phi) - V(\pm v) \\ =& -\frac{1}{2}|m|^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{3|m|^4}{2\lambda} \\ =& \frac{\lambda}{24}\left(\phi^2 - \frac{6|m|^2}{\lambda}\right)^2\,. \end{aligned} \tag{S.5}$$
The new potential $V'(\phi)$ is depicted in Figure 1.

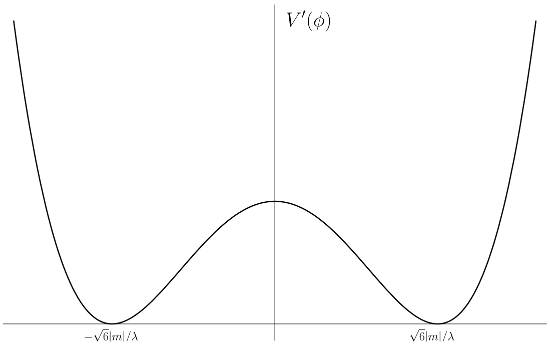

Figure 1: Scalar potential with two degenerate minima.

2. Given the potential of eq. (S.5), we look for static solutions, $\partial_t \phi = 0$, of the equation of motion (S.1)

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial V'}{\partial \phi} = 0\,. \tag{S.6}$$

The solution of such differential equation can be determined by multiplying each side by $\partial_x \phi$. In fact, we observe that

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2}\frac{\partial \phi}{\partial x} &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)^2 , \\ \frac{\partial V'}{\partial \phi}\frac{\partial \phi}{\partial x} &= \frac{\partial V'}{\partial x} \end{aligned} \tag{S.7}$$

and, hence, eq. (S.6) becomes

$$\frac{\partial}{\partial x}\left[\frac{1}{2}(\partial_x \phi)^2 - V'(\phi)\right] = 0\,, \tag{S.8}$$

which implies

$$\frac{1}{2}(\partial_x \phi)^2 - V'(\phi) = \text{const.} \tag{S.9}$$

The arbitrary constant is fixed by demanding that, asymptotically, the static solution corresponds to the constant solutions previously determined, $|\phi| \sim v$ for $|x| \to \infty$: since $V'(\pm v) = 0$, the constat that appear on the r.h.s of eq. (S.9) must be zero. Thus, we have

$$\partial_x \phi = \pm\sqrt{2V'(\phi)}\,. \tag{S.10}$$

By taking into account the explicit expression of the potential, we find

$$\partial_x \phi = \pm\frac{\sqrt{\lambda}}{2\sqrt{3}}(v^2 - \phi^2)\,. \tag{S.11}$$

Note that the two signs of the r.h.s., which determine the monotony property of ϕ, correspond to the two solution which interpolate, respectively, between $-v$ (v) and v ($-v$). By rescaling $\phi = v\tilde{\phi}$, we turn eq. (S.11) into

$$\partial_x \tilde{\phi} = \pm\frac{\sqrt{\lambda}v}{2\sqrt{3}}(1 - \tilde{\phi}^2)\,, \tag{S.12}$$

which can be easily integrated by separation of variables,

$$\frac{\mathrm{d}\tilde{\phi}}{(\tilde{\phi}^2-1)} = \pm\frac{\sqrt{\lambda}v}{2\sqrt{3}}\mathrm{d}x\,,$$

$$\int\left(\frac{\mathrm{d}\tilde{\phi}}{\tilde{\phi}+1} - \frac{\mathrm{d}\tilde{\phi}}{\tilde{\phi}-1}\right) = \pm\int\frac{\lambda v}{\sqrt{3}}\mathrm{d}x\,. \tag{S.13}$$

In this way, we obtain

$$\log\left(\frac{\tilde{\phi}+1}{\tilde{\phi}-1}\right) = \pm\frac{\sqrt{\lambda}v}{\sqrt{3}}x\,. \tag{S.14}$$

By using the inverse hyperbolic function $\operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$, we have

$$\tilde{\phi}_\pm = \pm\tanh\left(\frac{\sqrt{\lambda}v}{2\sqrt{3}}x\right) = \pm\tanh\left(\frac{|m|}{\sqrt{2}}x\right)\,, \tag{S.15}$$

or, in terms of the unnormalised field ϕ,

$$\phi_\pm = \pm\frac{\sqrt{6}|m|}{\sqrt{\lambda}}\tanh\left(\frac{|m|}{\sqrt{2}}x\right)\,. \tag{S.16}$$

Notice that the two solutions $\phi_\pm$ have the correct asymptotic limits for $x \to \pm\infty$.

3. In the static case, the Hamiltonian density is given by

$$\mathscr{H} = -\mathscr{L}\,, \tag{S.17}$$

and therefore, the energy of the soliton solutions is

$$E_\pm = \int_{-\infty}^{\infty}\mathrm{d}x\mathscr{H}(\phi_\pm) = \int_{-\infty}^{\infty}\mathrm{d}x\left[\frac{1}{2}(\partial_x\phi_\pm)^2 + V'(\phi_\pm)\right] \tag{S.18}$$

From the expression of $\phi_\pm$ given in eq. (S.16), we find

$$\frac{1}{2}(\partial_x\phi_\pm)^2 = V'(\phi_\pm) = \frac{3|m|^4}{2\lambda}\cosh^{-4}\left(\frac{|m|}{\sqrt{2}}\right) \tag{S.19}$$

and, hence,

$$\begin{aligned} E_\pm =& \frac{3|m|^4}{\lambda}\int_{-\infty}^{\infty}\mathrm{d}x\cosh^{-4}\left(\frac{|m|}{\sqrt{2}}\right) \\ =& \frac{3\sqrt{2}|m|^3}{\lambda}\int_{-\infty}^{\infty}\mathrm{d}y\cosh^{-4}y \\ =& \frac{4\sqrt{2}|m|^3}{\lambda}\,. \end{aligned} \tag{S.20}$$

4. We consider the normalised current

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu}\partial_\nu\tilde{\phi}\,, \quad \tilde{\phi} = \phi/v\,. \tag{S.21}$$

J^μ is obviously conserved, $\partial_\mu J^\mu = 0$, due to the antisymmetry of $\epsilon^{\mu\nu}$. The conserved charge is given by

$$\int_{-\infty}^{\infty}\mathrm{d}xJ^0 = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\partial_x\tilde{\phi} = \frac{1}{2}\,\tilde{\phi}\Big|_{-\infty}^{\infty} = 0, \pm 1\,, \tag{S.22}$$

where the values ± 1 correspond to the soliton solutions $\phi_\pm$, which interpolate between the two minima $\pm v$, while 0 correspond to the trivial vacuum solutions.