Kern- und Teilchenphysik I
Lecture 2: Fermi’s golden rule

(adapted from the Handout of Prof. Mark Thomson)

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http://www.physik.uzh.ch/de/lehrePHY211/HS2016.html
Lifetime

The probability that a particle survive a time $t + dt$ is given by $P(t + dt) = P(t)(1 - \Gamma dt)$, where $\Gamma$ is the decay probability per unit of time. We are basically saying that the probability that a particle decay at each time does not depend on the history of the particle.

\[
P(t + dt) = P(t)(1 - \Gamma dt) \rightarrow P(t + dt) - P(t) = P(t) \times (-\Gamma dt)
\]

\[
\frac{dP}{dt} = -\Gamma P \rightarrow P(t) = \alpha e^{-\Gamma t}
\]

In order to find the coefficient $\alpha$ we can just impose $P(0) = 1$, i.e. the particle (by definition) exists at $t=0$, so we get $P(t) = e^{-\Gamma t}$

We can derive the Probability density Function (PdF) such that $\int_0^\infty P(t)dt = 1$ which implies $A \int_0^\infty e^{-\Gamma t}dt = 1 \rightarrow A \left[-\frac{1}{\Gamma} e^{-\Gamma t}\right]_0^\infty = A\frac{1}{\Gamma} = 1 \rightarrow P(t) = \frac{1}{\Gamma} e^{-\Gamma t}$

Finally if we compute the expectation value of $t$ we get $\langle t \rangle = \tau = \int_0^\infty P(t)dt = \frac{1}{\Gamma}$

In a general frame $P(t) = e^{-\frac{t}{\tau}}$ and $\mathcal{P}(t) = \tau e^{-\frac{t}{\tau}}$
The decay width measures the probability per time units. Its units are GeV.

Physical meaning:

\[ \Delta E \Delta t \geq \hbar \]

- The mass of the particle we quote is the position of the peak.
- Particles with very short lifetime have a large width, particles with a long lifetime have a narrow width.
- Except for particles decaying strongly we cannot usually resolve the width.
Let’s take a non stable particle \( \psi(t) = \psi(0)e^{-iEt-\Gamma/2t} \rightarrow |\psi(t)|^2 \propto e^{-\Gamma t} \)

An unstable particle can be describe by having an imaginary part \((\Gamma/2)\) in the energy

If the particle is unstable it means it is not an eigenstate of the Hamiltonian (indeed it has a small imaginary part of the energy). Let’s write this state as a function of the energy eigenstates:

\[
\tilde{\alpha}(E) = \int \psi(0)e^{-iMt+\frac{\Gamma}{2}t}e^{iEt} \propto \frac{1}{E-M+(i\frac{\Gamma}{2})}
\]

\[
|\tilde{\alpha}|^2 \propto \frac{1}{(E-M)^2+(\frac{\Gamma}{2})^2}
\]

More rigorous arguments lead to the same result
The "cross section", $\sigma$, can be thought of as the effective cross-sectional area of the target particles for the interaction to occur.

In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption.

Here $\sigma$ is the projective area of nucleus.

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

$$d\Omega = d(\cos \theta)d\phi$$

with

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$
Cross Section

In time $\delta t$, a particle of type $a$ traverses region containing $n_b(v_a + v_b)A\delta t$ particles of type $b$.

Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a + v_b)A\delta t$ particles of type $b$.

Interaction Probability =

$\frac{n_b(v_a + v_b)A\delta t \sigma}{A} = n_b v\delta t \sigma$

$[v = v_a + v_b]$

Rate per particle of type $a = n_b v \sigma$

Consider volume $V$, total reaction rate = $(n_b v \sigma).(n_a V) = (n_b V)(n_a v) \sigma$

= $N_b \phi_a \sigma$

As anticipated: Rate = Flux x Number of targets x cross section
Luminosity

We define the luminosity such that:

\[ W = \frac{dP}{dt} = \mathcal{L} \cdot \sigma \]

\[ [\mathcal{L}] = \left[ \frac{1}{L^2 T} \right] \]

We often see the integrated luminosity as

\[ L = \int \mathcal{L} \rightarrow P = \int W dt = L \sigma \]

\[ W = N_b \phi_a \sigma \rightarrow \mathcal{L} = N_b \phi_a \]

\[ \phi = \frac{\text{#particles}}{At} \]
Consider to shoot particles at an infinitesimal slab

The number of interactions is given by \( n_b v \sigma dt = n_b \sigma dx = N_A \rho_b \sigma dx \)

Mean Free Path: \( \ell = \frac{1}{n_b \sigma} \)

If the scattering centers are nucleons \( \ell = \frac{1}{N_A \rho_b \sigma} \)

where \( N_A \) is the Avogadro number

The particle has the same probability of interacting anywhere, i.e.: \( P(x+dx) = P(x)(1 - n_b \sigma dx) \rightarrow P(x) = \alpha e^{-n_b \sigma x} = \alpha e^{-\frac{x}{\ell}} \)
Units for cross-section

The cross-section is often measured in barn $(b)$

$$1b = (10 \text{ fm}) \times (10 \text{ fm}) = 100 \text{ fm}^2 = 10^{-28}m^2 = 10^{-24}cm^2$$

E.g. Total $pp$ cross section at $7\text{TeV} \sim 90\text{mb}$
compute number of interactions in 1 year running at the luminosity of

$$\mathcal{L} = 2 \times 10^{32}cm^{-2}s^{-1}$$

$$L = \int \mathcal{L}dt = 2 \times 10^{32}cm^{-2}s^{-1} \times \pi \cdot 10^7s =
= 2 \times 10^{39}cm^{-2} = 2 \times 10^{15}\frac{1}{b} = 2f\text{b}^{-1} = 2 \times 10^{12}\text{mb}^{-1}$$

$$N = L \times \sigma = 2 \times 10^{12} \times 90 \times \frac{mb}{mb} = 1.8 \times 10^{14}$$
In particle physics we are mainly concerned with particle interactions and decays, i.e. transitions between states.

- These are the experimental observables of particle physics.

Calculate transition rates from Fermi’s Golden Rule:

\[ \Gamma_{fi} = 2\pi|T_{fi}|^2 \rho(E_f) \]

- \( \Gamma_{fi} \) is number of transitions per unit time from initial state \( |i\rangle \) to final state \( |f\rangle \) – not Lorentz Invariant!
- \( T_{fi} \) is Transition Matrix Element

\[ T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots \]

- \( \hat{H} \) is the perturbing Hamiltonian
- \( \rho(E_f) \) is density of final states

★ Rates depend on MATRIX ELEMENT and DENSITY OF STATES.

The ME contains the fundamental particle physics. Just kinematics.
Aiming towards a proper calculation of decay and scattering processes:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $\mu^-q \rightarrow e^- q$

Need relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times \text{(phase space)}$$

Need relativistic treatment of spin-half particles:

- Dirac Equation

Need relativistic calculation of interaction Matrix Element:

- Interaction by particle exchange and Feynman rules
- and a few mathematical tricks along, e.g. the Dirac Delta Function
Decay rate and Cross Section

• Consider the two-body decay 
  \[ i \rightarrow 1 + 2 \]

• Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):
  \[
  \psi_1 = N e^{i(\vec{p}.\vec{r} - E t)} \\
  = N e^{-i p.x}
  \]
  where \( N \) is the normalisation and \( p.x = p^\mu x_\mu \)

For decay rate calculation need to know:

- Wave-function normalisation
- Transition matrix element from perturbation theory
- Expression for the density of states

★ First consider wave-function normalisation
  - Previously (e.g. part II) have used a non-relativistic formulation
  - Non-relativistic: normalised to one particle in a cube of side \( a \)
  \[
  \int \psi \psi^* dV = N^2 a^3 = 1 \quad \Rightarrow \quad N^2 = 1/a^3
  \]
Non-relativistic Phase Space (revision)

- Volume of single state in momentum space:
  \[ \left( \frac{2\pi}{a} \right)^3 = \frac{(2\pi)^3}{V} \]

- Normalising to one particle/unit volume gives number of states in element:
  \[ \frac{d^3 \vec{p}}{(2\pi)^3} \times \frac{1}{V} = \frac{d^3 \vec{p}}{(2\pi)^3} \]

- Therefore density of states in Golden rule:
  \[ \rho(E_f) = \left| \frac{d^n}{dE} \right|_{E_f} = \left| \frac{dn}{d|\vec{p}|} \frac{d|\vec{p}|}{dE} \right|_{E_f} \]

- Integrating over an elemental shell in momentum-space gives
  \[ (d^3 \vec{p} = 4\pi p^2 dp) \]
  \[ \rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \frac{1}{\beta} \]

with
\[ E^2 = p^2 + m^2 \]
\[ 2E \, dE = 2p \, dp \]
\[ \frac{dp}{dE} = \frac{E}{p} = \frac{1}{\beta} \]
• In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: “infinitely narrow spike of unit area”

$$\delta(x - a)$$

\[ \int_{-\infty}^{+\infty} \delta(x - a) \, dx = 1 \]

\[ \int_{-\infty}^{+\infty} f(x) \delta(x - a) \, dx = f(a) \]

• Any function with the above properties can represent $\delta(x)$

\[ \delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \]

(An infinitesimally narrow Gaussian)

• In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \to 1 + 2$

\[ \int \ldots \delta(E_a - E_1 - E_2) \, dE \quad \text{and} \quad \int \ldots \delta^3(p_a - p_1 - p_2) \, d^3 p \]

express energy and momentum conservation
We will soon need an expression for the delta function of a function \( \delta(f(x)) \)

- Start from the definition of a delta function
  \[
  \int_{y_1}^{y_2} \delta(y) \, dy = \begin{cases} 
  1 & \text{if } y_1 < 0 < y_2 \\
  0 & \text{otherwise}
  \end{cases}
  \]

- Now express in terms of \( f(x) \) where \( f(x_0) = 0 \)
  \[
  \int_{x_1}^{x_2} \delta(f(x)) \frac{df}{dx} \, dx = \begin{cases} 
  1 & \text{if } x_1 < x_0 < x_2 \\
  0 & \text{otherwise}
  \end{cases}
  \]

- From properties of the delta function (i.e. here only non-zero at \( x_0 \))
  \[
  \left| \frac{df}{dx} \right|_{x_0} \int_{x_1}^{x_2} \delta(f(x)) \, dx = \begin{cases} 
  1 & \text{if } x_1 < x_0 < x_2 \\
  0 & \text{otherwise}
  \end{cases}
  \]

- Rearranging and expressing the RHS as a delta function
  \[
  \int_{x_1}^{x_2} \delta(f(x)) \, dx = \frac{1}{\left| \frac{df}{dx} \right|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0) \, dx
  \]

\[
\delta(f(x)) = \left| \frac{df}{dx} \right|^{-1}_{x_0} \delta(x - x_0)
\]

(1)
Fermi’s Golden Rule

\[ \Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f) \]

• Rewrite the expression for density of states using a delta-function

\[ \rho(E_f) = \left| \frac{\text{d}n}{\text{d}E} \right|_{E_f} = \int \frac{\text{d}n}{\text{d}E} \delta(E - E_i) \text{d}E \]

since \( E_f = E_i \)

Note: integrating over all final state energies but energy conservation now taken into account explicitly by delta function

• Hence the golden rule becomes:

\[ \Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E) \text{d}n \]

the integral is over all “allowed” final states of any energy

• For \( d\text{n} \) in a two-body decay, only need to consider one particle: mom. conservation fixes the other

\[ \Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \frac{d^3 \vec{p}_1}{(2\pi)^3} \]

• However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another \( \delta \)-fn

\[ \Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3 \vec{p}_1}{(2\pi)^3} \frac{d^3 \vec{p}_2}{(2\pi)^3} \]

Energy cons. \quad Mom. cons. \quad Density of states
Lorentz Invariant PH

- In non-relativistic QM normalise to one particle/unit volume: \( \int \psi^* \psi \, dV = 1 \)
- When considering relativistic effects, volume contracts by \( \gamma = \frac{E}{m} \)

\[ \begin{array}{c}
\text{a} \\
\text{a} \\
\text{a}
\end{array} \quad \begin{array}{c}
\text{a} \\
\text{a} \\
\text{a}/\gamma
\end{array} \]

- Particle density therefore increases by \( \gamma = \frac{E}{m} \)

\[ \star \text{Conclude that a relativistic invariant wave-function normalisation needs to be proportional to } E \text{ particles per unit volume} \]

- Usual convention: Normalise to 2E particles/unit volume
  \( \int \psi'^* \psi' \, dV = 2E \)
  \( \int \psi^* \psi \, dV = 1 \)

- Previously used \( \psi \) normalised to 1 particle per unit volume
- Hence \( \psi' = (2E)^{1/2} \psi \) is normalised to 2E per unit volume

- Define Lorentz Invariant Matrix Element, \( M_{fi} \), in terms of the wave-functions normalised to 2E particles per unit volume

\[
M_{fi} = \langle \psi'_1 \psi'_2 \ldots | \hat{H} | \ldots \psi'_{n-1} \psi'_n \rangle = (2E_1.2E_2.2E_3 \ldots 2E_n)^{1/2} T_{fi}
\]
• We have that $\psi' = \sqrt{2E}\psi$, therefore $|\psi'|^2 = 2E|\psi|^2$

• Let’s start in the rest frame $\int |\psi'|^2 dV_0 = 2E \int |\psi|^2 dV_0 = 2M \int |\psi|^2 dV_0 = 2E \times 1$

• Now in another frame $E = M \rightarrow E = M\gamma$ while $V_0 \rightarrow \frac{V_0}{\gamma}$

• We have $\int |\psi'|^2 dV_0 = 2M \int |\psi|^2 dV_0 \rightarrow 2\gamma M \int |\psi|^2 \frac{dV_0}{\gamma}$
Fermi’s Golden Rule

• For the two body decay $i \rightarrow 1 + 2$

$M_{fi} = \langle \psi'_1 \psi'_2 | \hat{H}' | \psi'_i \rangle$

$= (2E_i.2E_1.2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle$

$= (2E_i.2E_1.2E_2)^{1/2} T_{fi}$

★ Now expressing $T_{fi}$ in terms of $M_{fi}$ gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3 \vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3 \vec{p}_2}{(2\pi)^3 2E_2}$$

Note:

• $M_{fi}$ uses relativistically normalised wave-functions. It is Lorentz Invariant

• $\frac{d^3 \vec{p}}{(2\pi)^3 2E}$ is the Lorentz Invariant Phase Space for each final state particle

• the factor of $2E$ arises from the wave-function normalisation

(prove this in Question 2)

• This form of $\Gamma_{fi}$ is simply a rearrangement of the original equation

but the integral is now frame independent (i.e. L.I.)

• $\Gamma_{fi}$ is inversely proportional to $E_i$, the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_i = \gamma m$).

• Energy and momentum conservation in the delta functions
- Let’s consider the total Hamiltonian $H = H_0 + H'$

$$H_0 |\phi_k\rangle = E_k |\phi_k\rangle \quad \langle \phi_k | \phi_j \rangle = \delta_{kj}$$

$\phi_k$ are the eigenstates of the unperturbed Hamiltonian ($H_0$)

$$i \frac{d\psi(t)}{dt} = [H_0 + H'(t)] \psi$$

- We can still write a generic wave function as a function of the eigenstates of the unperturbed Hamiltonian

- This time the coefficient will depend on time, since the eigenstates of $H_0$ are not stationary

$$\psi(x, t) = \sum_k c_k(t) \phi_k(x) e^{-iE_k t}$$
Fermi’s Golden Rule

- Let’s consider the total Hamiltonian $H = H_0 + H'$

$$H_0 \phi_k = E_k \phi_k \quad \quad \langle \phi_k | \phi_j \rangle = \delta_{k,j}$$

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- We can still write a generic wave function as a function of the eigenstates of the unperturbed Hamiltonian
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$$\psi(x, t) = \sum_k c_k(t) \phi_k(x) e^{-iE_k t}$$

Eigenstate of $H_0$ at time $t$

Coefficients depend on time
Let's now put $\psi(x, t)$ into the Schrödinger equation:

$$i \sum_k \left( \frac{dc_k(t)}{dt} \phi_k e^{-iE_k t} - iE_k c_k(t) \phi_k e^{-iE_k t} \right) = \sum_k c_k(t) H_0 \phi_k e^{-iE_k t} + \sum_k c_k(t) H' \phi_k e^{-iE_k t}$$

$$\sum_k E_k c_k(t) \phi_k e^{-iE_k t}$$

Let's now assume that $|i\rangle = \phi_i$ (e.g. excited state of Hydrogen atom) and that $H'$ is constant for $t > 0$. If $H'$ is sufficiently small $c_i(t) \approx 1$ and $c_k \neq i(t) \approx 0$

$$i \sum_k \frac{dc_k(t)}{dt} \phi_k e^{-iE_k t} \approx H' \phi_i e^{-iE_i t},$$

where $c_k \neq i \approx 0$ was used on the left side.
**Fermi’s Golden Rule**

\[
i \sum_k \frac{dc_k(t)}{dt} \phi_k e^{-iE_k t} \sim H' \phi_i e^{-iE_i t}
\]

We want to compute the transition probability to the final state \( |f\rangle \), so I multiply by \( \langle f | = \langle \phi_f | \) (e.g. transition to the ground state of the Hydrogen atom)

\[
\frac{dc_f(t)}{dt} = -i \langle f | H' | i \rangle e^{i(E_f - E_i) t}
\]

Now \( \langle f | H' | i \rangle = T_{if} = \int_V \phi_f^* H' \phi_i dV \)

\[
c_f = -iT_{if} \int_{-T/2}^{T/2} e^{i(E_f - E_i) t} dt
\]

Assuming \( H' \) is time independent in \([ -T/2, T/2 ]\)

Let's calculate the transition probability:

\[
P_{fi} = c_f c_f^* = |T_{if}|^2 \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} e^{i(E_f - E_i) t} e^{i(E_f - E_i) \tau} dt d\tau
\]
Fermi’s Golden Rule

The probability per unit of time is given by:

\[ d\Gamma_{if} = \frac{P_{fi}}{T} = \frac{1}{T} |T_{if}|^2 \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} e^{i(E_f - E_i)t} e^{i(E_f - E_i)\tau} dtd\tau \]

Remember the definition of the Dirac’s delta: \( \delta(x) = \frac{1}{2\pi} \int e^{i(kx)}dk \) we get:

\[ d\Gamma_{if} = 2\pi |T_{if}|^2 \frac{1}{T} \int_{-T/2}^{T/2} e^{-i(E_f - E_i)\tau} \delta(E_f - E_i)d\tau \]

You might be surprised that we get the \( \delta(E_f - E_i) \), this means that this result to be used needs an integration over \( E \), why? The reason is that we are computing the probability for a transition to the final state energy in the range \([E_f, E_f + dE_f]\) we need to sum over all possibilities and the \( \delta \) will ensure energy is conserved.

\[ \Gamma_{fi} = 2\pi |T_{if}|^2 \int dn_{E_f} \delta(E_f - E_i) \left( \frac{1}{T} \int_{-T/2}^{T/2} d\tau \right) = 1 \text{ (when } E_f = E_i) \]

\[ dn_{E_f} = \frac{dn}{dE_f} dE_f \]
Fermi’s Golden Rule

finally we get the Fermi’s golden rule:

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \left| \frac{dn}{dE_f} \right|_{E_i} = 2\pi |T_{fi}|^2 \rho(E_i)$$

- At first order we had $T_{fi} = \langle f|H'|i \rangle$, where we had assumed $c_k \neq i \simeq 0$

- We can calculate the second order by using an iterative procedure, by substituting $c_f(T) = -iT_{fi} \int e^{i(E_f-E_i)t} dt$ into $i \sum_k \frac{dc_k}{dt} \phi_k e^{-iE_k t} = \sum_k H' c_k(t) \phi_k e^{-iE_k t}$

- We would get $T_{fi} = \langle f|H'|i \rangle + \sum_{k \neq i} \frac{\langle f|H' |k \rangle \langle k|H'|i \rangle}{E_i - E_k}$

Detailed calculation in the book by Prof. Thomson (Sec. 2.3.6)
Decay rate calculation

\[ \Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3 \vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3 \vec{p}_2}{(2\pi)^3 2E_2} \]

★ Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient

- In the C.o.M. frame \( E_i = m_i \) and \( \vec{p}_i = 0 \)

\[ \Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3 \vec{p}_1}{2E_1} \frac{d^3 \vec{p}_2}{2E_2} \]

- Integrating over \( \vec{p}_2 \) using the \( \delta \)-function:

\[ \Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3 \vec{p}_1}{4E_1 E_2} \]

now \( E_2^2 = (m_2^2 + |\vec{p}_1|^2) \) since the \( \delta \)-function imposes \( \vec{p}_2 = -\vec{p}_1 \)

- Writing \( d^3 \vec{p}_1 = p_1^2 dp_1 \sin \theta d\theta d\phi = p_1^2 dp_1 d\Omega \)

\[ \Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta \left( m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2} \right) \frac{p_1^2 dp_1 d\Omega}{E_1 E_2} \]
Decay Rate Calculation

- Which can be written in the form
  \[ \Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1) \delta(f(p_1)) dp_1 d\Omega \]  \hspace{1cm} (2)

  where \[ g(p_1) = \frac{p_1^2}{(E_1 E_2)} = \frac{p_1^2}{(m_1^2 + p_1^2)^{1/2}(m_2^2 + p_1^2)^{1/2}} \]

  and \[ f(p_1) = m_i - (m_1^2 + p_1^2)^{1/2} - (m_2^2 + p_1^2)^{1/2} \]

  Note:
  - \( \delta(f(p_1)) \) imposes energy conservation.
  - \( f(p_1) = 0 \) determines the C.o.M momenta of the two decay products
  - i.e. \( f(p_1) = 0 \) for \( p_1 = p^* \)

- Eq. (2) can be integrated using the property of \( \delta \)–function derived earlier (eq. (1))
  \[ \int g(p_1) \delta(f(p_1)) dp_1 = \frac{1}{|df/dp_1|_{p^*}} \int g(p_1) \delta(p_1 - p^*) dp_1 = \frac{g(p^*)}{|df/dp_1|_{p^*}} \]

  where \( p^* \) is the value for which \( f(p^*) = 0 \)

- All that remains is to evaluate \( df/dp_1 \)
  \[ \frac{df}{dp_1} = -\frac{p_1}{(m_1^2 + p_1^2)^{1/2}} - \frac{p_1}{(m_2^2 + p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1 + E_2}{E_1 E_2} \]
Decay Rate Calculation

\[
\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{E_1 E_2}{p_1 (E_1 + E_2)} \frac{p_1^2}{E_1 E_2} \right|_{p_1 = p^*} \mathrm{d}\Omega
\]

\[
= \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1 = p^*} \mathrm{d}\Omega
\]

- But from \( f(p_1) = 0 \), i.e. energy conservation: \( E_1 + E_2 = m_i \)

\[
\Gamma_{fi} = \frac{\vec{p}^*}{32\pi^2 E_i m_i} \int |M_{fi}|^2 \mathrm{d}\Omega
\]

In the particle’ s rest frame \( E_i = m_i \)

\[
\frac{1}{\tau} = \Gamma = \frac{\vec{p}^*}{32\pi^2 m_i^2} \int |M_{fi}|^2 \mathrm{d}\Omega
\]

(3)

VALID FOR ALL TWO-BODY DECAYS !

- \( p^* \) can be obtained from \( f(p_1) = 0 \)

\[
(m_1^2 + p^{*2})^{1/2} + (m_2^2 + p^{*2})^{1/2} = m_i
\]

\[
p^* = \frac{1}{2m_i} \sqrt{[(m_i^2 - (m_1 + m_2)^2) [m_i^2 - (m_1 - m_2)^2]]}
\]
Mandelstam variables

★ In particle scattering/annihilation there are three particularly useful Lorentz Invariant quantities: $s$, $t$ and $u$

★ Consider the scattering process $1 + 2 \rightarrow 3 + 4$

• Can define three kinematic variables: $s$, $t$ and $u$ from the following four vector scalar products (squared four-momentum of exchanged particle)

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Note:

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

• This is a scalar product of two four-vectors Lorentz Invariant

• Since this is a L.I. quantity, can evaluate in any frame. Choose the most convenient, i.e. the centre-of-mass frame:

$$p_1^* = (E_1^*, \vec{p}_1^*) \quad p_2^* = (E_2^*, -\vec{p}_2^*)$$

$$s = (E_1^* + E_2^*)^2$$

is the total energy of collision in the centre-of-mass frame
Cross section calculation

- Consider scattering process
  \[ 1 + 2 \rightarrow 3 + 4 \]

- Start from Fermi’s Golden Rule:
  \[
  \Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{d^3\vec{p}_4}{(2\pi)^3}
  \]

  where \( T_{fi} \) is the transition matrix for a normalisation of 1/unit volume

- Now
  \[
  \text{Rate/Volume} = (\text{flux of } 1) \times (\text{number density of } 2) \times \sigma
  \]
  \[
  = n_1(v_1 + v_2) \times n_2 \times \sigma
  \]

- For 1 target particle per unit volume
  \[
  \text{Rate} = (v_1 + v_2)\sigma
  \]

  \[
  \sigma = \frac{\Gamma_{fi}}{v_1 + v_2}
  \]

  \[
  \sigma = \frac{(2\pi)^4}{v_1 + v_2} \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{d^3\vec{p}_4}{(2\pi)^3}
  \]

  the parts are not Lorentz Invariant
Cross section calculation

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element

$$M_{fi} = (2E_1 2E_2 2E_3 2E_4)^{1/2}T_{fi}$$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (\nu_1 + \nu_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form

- The quantity $F = 2E_1 2E_2 (\nu_1 + \nu_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant

$$F = 4 \left[ (p_1^\mu p_2^\mu)^2 - m_1^2 m_2^2 \right]^{1/2}$$

- Consequently cross section is a Lorentz Invariant quantity

Two special cases of Lorentz Invariant Flux:

- **Centre-of-Mass Frame**

  $$F = 4E_1 E_2 (\nu_1 + \nu_2)$$

  $$= 4E_1 E_2 (|\vec{p}^*|/E_1 + |\vec{p}^*|/E_2)$$

  $$= 4|\vec{p}^*| (E_1 + E_2)$$

  $$= 4|\vec{p}^*| \sqrt{s}$$

- **Target (particle 2) at rest**

  $$F = 4E_1 E_2 (\nu_1 + \nu_2)$$

  $$= 4E_1 m_2 \nu_1$$

  $$= 4E_1 m_2 (|\vec{p}_1|/E_1)$$

  $$= 4m_2 |\vec{p}_1|$$
Cross section calculation

• We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame

• Start from

\[
\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3 \vec{p}_3}{2E_3} \frac{d^3 \vec{p}_4}{2E_4}
\]

• Here \( \vec{p}_1 + \vec{p}_2 = 0 \) and \( E_1 + E_2 = \sqrt{s} \)

\[\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \frac{d^3 \vec{p}_3}{2E_3} \frac{d^3 \vec{p}_4}{2E_4}\]

★ The integral is exactly the same integral that appeared in the particle decay calculation but with \( m_a \) replaced by \( \sqrt{s} \)

\[\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*\]

\[\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*\]
Appendix

In case you want to have fun with other scattering processes you can look at the appendix
In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 d\Omega^*$$

For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 d\Omega^*$$

because the angles in $d\Omega^* = d(\cos \theta^*) d\phi^*$ refer to the C.o.M frame.

For the last calculation in this section, we need to find a L.I. expression for $d\sigma$

★ Start by expressing $d\Omega^*$ in terms of Mandelstam $t$ i.e. the square of the four-momentum transfer

$$t = q^2 = (p_1 - p_3)^2$$
Cross section calculation

• Want to express \( d\Omega^* \) in terms of Lorentz Invariant \( dt \)

  where \( t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3 \)

• In C.o.M. frame:

  \[
  \begin{align*}
  p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\
  p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*| \sin \theta^*, 0, |\vec{p}_3^*| \cos \theta^*) \\
  p_1^\mu p_3^\mu &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*| \cos \theta^* \\
  t &= m_1^2 + m_3^2 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*| \cos \theta^*
  \end{align*}
  \]

  giving

  \[ dt = 2|\vec{p}_1^*||\vec{p}_3^*|d(\cos \theta^*) \]

  therefore

  \[ d\Omega^* = d(\cos \theta^*)d\phi^* = \frac{dt d\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|} \]

  hence

  \[ d\sigma = \frac{1}{64\pi^2 s |\vec{p}_1^*|^2} |M_{fi}|^2 d\Omega^* = \frac{1}{2 \cdot 64\pi^2 s |\vec{p}_1^*|^2} |M_{fi}|^2 d\phi^* dt \]

• Finally, integrating over \( d\phi^* \) (assuming no \( \phi^* \) dependence of \( |M_{fi}|^2 \)) gives:

\[
\frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{p}_1^*|^2} |M_{fi}|^2
\]
LI differential cross section

- All quantities in the expression for $\frac{d\sigma}{dt}$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation:

$$|\vec{p}_i^*|^2 = \frac{1}{4s} [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$$

- As an example of how to use the invariant expression $\frac{d\sigma}{dt}$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$.

In this limit

$$|\vec{p}_i^*|^2 = \frac{(s - m_2)^2}{4s}$$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi(s - m_2^2)^2}|M_{fi}|^2$$

$(m_1 = 0)$
2->2 Body scattering in the Lab

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering).

- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: \( m_1 = m_3 = 0, \quad m_2 = m_4 = M \)

\[
(E_1, |\vec{p}_1|) \quad (E_2, |\vec{p}_2|) \quad (E_3, |\vec{p}_3|) \quad (E_4, |\vec{p}_4|)
\]

\[\theta\]

\[\text{e.g.} \quad e^- \quad X \quad e^- \]

- Wish to express the cross section in terms of scattering angle of the \( e^- \)

\[
d\Omega = 2\pi d(\cos \theta)
\]

therefore

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt} \frac{dt}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos \theta)} \frac{d\sigma}{dt}
\]

- The rest is some rather tedious algebra…. start from four-momenta

\[
p_1 = (E_1, 0, 0, E_1), \quad p_2 = (M, 0, 0, 0), \quad p_3 = (E_3, E_3 \sin \theta, 0, E_3 \cos \theta), \quad p_4 = (E_4, \vec{p}_4)
\]

so here

\[t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1 - \cos \theta)\]

But from (E,p) conservation

\[p_1 + p_2 = p_3 + p_4\]

and, therefore, can also express \( t \) in terms of particles 2 and 4
2->2 Body scattering in the Lab

\[ t = (p_2 - p_4)^2 = 2M^2 - 2p_2 \cdot p_4 = 2M^2 - 2ME_4 \]
\[ = 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3) \]

Note \( E_1 \) is a constant (the energy of the incoming particle) so

\[ \frac{dt}{d(\cos \theta)} = 2M \frac{dE_3}{d(\cos \theta)} \]

• Equating the two expressions for \( t \) gives

\[ E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos \theta} \]

so

\[ \frac{dE_3}{d(\cos \theta)} = \frac{E_1 M}{(M + E_1 - E_1 \cos \theta)^2} = E_1^2 M \left( \frac{E_3}{E_1 M} \right)^2 = \frac{E_3^2}{M} \]

\[ \frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos \theta)} \frac{d\sigma}{dt} = \frac{1}{2\pi} \frac{2M E_3^2}{M} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{1}{16\pi(s - M^2)^2} |M_{fi}|^2 \]

using \( s = (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 = M^2 + 2ME_1 \)

\[ (s - M^2) = 2ME_1 \]

\[ \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left( \frac{E_3}{ME_1} \right)^2 |M_{fi}|^2 \]

Particle 1 massless \( \rightarrow (p_1^2 = 0) \)

In limit \( m_1 \rightarrow 0 \)
In this equation, $E_3$ is a function of $\theta$:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos \theta}$$

giving

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left( \frac{1}{M + E_1 - E_1 \cos \theta} \right)^2 |M_{fi}|^2$$

$(m_1 = 0)$

General form for 2→2 Body Scattering in Lab. Frame

The calculation of the differential cross section for the case where $m_1$ cannot be neglected is longer and contains no more “physics” It gives:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3 |\vec{p}_1| \cos \theta} \cdot |M_{fi}|^2$$

Again there is only one independent variable, $\theta$, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3| \cos \theta + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of $\theta$

$$\vec{p}_4 = \vec{p}_1 - \vec{p}_3$$