Exercise 1. *Weak Decay of the Pion*

1. Consider the Lagrangian for semileptonic weak interactions:

$$\mathscr{L} = \frac{4G_F}{\sqrt{2}} \left(\bar{\ell}_L \gamma^\mu \nu_L\right) \left(\bar{u}_L \gamma_\mu d_L\right) + \text{h.c.} \tag{1}$$

with $\nu_L = P_L \nu = 1/2(1-\gamma^5)\nu$. Using the quark currents defined as

$$J^{\mu a} = \bar{Q}\gamma^\mu \tau^a Q\,, \qquad J^{\mu 5 a} = \bar{Q}\gamma^\mu \gamma^5 \tau^a Q\,, \tag{2}$$

where $Q = (u\ d)^T$ is the quark doublet and $\tau^a = \sigma^a/2$ are the generators of $SU(2)$, show that

$$\bar{u}_L \gamma^\mu d_L = \frac{1}{2}\left(J^{\mu 1} + iJ^{\mu 2} - J^{\mu 51} - iJ^{\mu 52}\right). \tag{3}$$

2. The matrix element of $J^{\mu 5a}$ between the vacuum and an on-shell pion can be written as

$$\langle 0|J^{\mu 5a}|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ip\cdot x}\,, \tag{4}$$

where f_π is a constant with the dimension of a mass. Using this identification together with the result of part 1, show that the amplitude for the decay $\pi^+ \to \ell^+\nu$ ($|\pi^+\rangle = 1/\sqrt{2}(|\pi^1\rangle + i\,|\pi^2\rangle)$) is given by

$$\mathscr{M} = G_F f_\pi \bar{u}(q)\not{p}(1-\gamma^5)v(k)\,, \tag{5}$$

where p, k and q are the momenta of π^+, ℓ^+ and ν respectively.

3. Compute the decay rate of the pion. Show that this rate vanishes in the limit of zero lepton mass, and that the relative rate of pion decay to muons and electrons is given by

$$\frac{\Gamma(\pi^+ \to e^+\nu)}{\Gamma(\pi^+ \to \mu^+\nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \frac{\left(1 - m_e^2/m_\pi^2\right)^2}{\left(1 - m_\mu^2/m_\pi^2\right)^2} \approx 10^{-4}. \tag{6}$$

From the measured pion lifetime $\tau_\pi = 2.6 \cdot 10^{-8}$ s, the Fermi constant $G_F = 1.17 \cdot 10^{-5}$ GeV^{-2}, the conversion factor $1\,\text{s} = 1.52 \cdot 10^{21}$ MeV^{-1} and the pion and muon masses $m_\pi = 140$ MeV, $m_\mu = 106$ MeV, determine the value of f_π.

Exercise 2. *Pseudoscalar Higgs coupling to gluons*

We consider the production of a pseudoscalar Higgs A_0 in gluon fusion. $gg \to A_0$ is a process induced by a massive quark loop, since A_0 does not couple directly to gluons.

1. Verify that, at one-loop level, the amplitude can be written as

$$\mathscr{M} = \varepsilon(p_1)_\alpha \varepsilon(p_2)_\beta \left[\mathscr{M}_1^{\alpha\beta} + \mathscr{M}_2^{\alpha\beta}\right]\,, \tag{7}$$

where $\mathscr{M}_i^{\alpha\beta}$ are the contributions of two distinct Feynman diagrams. Knowing that the coupling of A_0 to quarks is given by $\frac{-i}{\sqrt{2}} y_f \delta_{ij}\gamma^5$ (where the Yukawa couling $y_f = \sqrt{2}m_f/v$

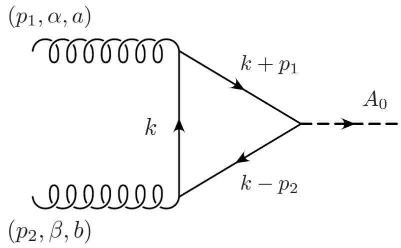

Figure 1: One of the two leading order diagrams for $gg \to A_0$.

is proportional to the mass of the fermion m_f and i, j are colour indices) show that the diagram depicted in Figure 1 corresponds to

$$\mathscr{M}_1^{\alpha\beta} = -\frac{y_f (g_s \mu^\epsilon)^2}{\sqrt{2}} \mathrm{Tr}[T^a T^b] \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^5 (\not{k} + \not{p}_1 + m_f)\gamma^\alpha (\not{k} + m_f)\gamma^\beta (\not{k} - \not{p}_2 + m_f)\right]}{((k+p_1)^2 - m_f^2)(k^2 - m_f^2)((k-p_2)^2 - m_f^2)} , \tag{8}$$

where g_s is the coupling of the strong interaction and T^a are colour matrices, which satisfy $\mathrm{Tr}[T^a T^b] = 1/2\, \delta^{ab}$. Evaluate the trace and relate the second diagram with the two gluons exchanged to the first one (Hint: since $\mathscr{M}_1^{\alpha\beta}$ turns out to be finite, you can safely work in four dimensions. Before computing the trace, use the symmetry properties of traces involving γ^5 in order to simplify the calculation).
After perfoming the trace, you will be left with the loop integral

$$I(p_1, -p_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{((k+p_1)^2 - m_f^2)(k^2 - m_f^2)((k-p_2)^2 - m_f^2)} = -\frac{i}{8\pi^2} \frac{1}{m_A^2} f(\tau) \tag{9}$$

with $\tau = \frac{4m_f^2}{m_A^2}$ and

$$f(\tau) = \begin{cases} \arcsin^2\left(\frac{1}{\sqrt{\tau}}\right) & \tau \geq 1 \\ \frac{-1}{4}\left(\log\left[\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}}\right] - i\pi\right)^2 & \tau < 1\,. \end{cases} \tag{10}$$

2. Compute the square matrix element and average it over colour and polarisations,

$$|\overline{\mathscr{M}}|^2 = \frac{1}{N_p^2 N_g^2} \sum_{\text{polarisations}} \left| \varepsilon_\alpha(p_1) \varepsilon_\beta(p_2) \mathscr{M}^{\alpha\beta} \right|^2 , \tag{11}$$

where $N_p = 2$ denotes the number of gluon polarisations and $N_g = 8$ is the number of possible values of the colour indices a and b. In the sum over polarisations, you can use

$$\sum_{\text{polarisations}} \varepsilon_\nu(k) \varepsilon_\mu^*(k) \to -g_{\mu\nu} ,$$

as in QED. Compute the total partonic cross section $\hat{\sigma}_{gg \to A_0}$.

3. The total cross section for $pp \to A_0$ can be written as

$$\sigma_{pp \to A_0} = \int \mathrm{d}x_1 \mathrm{d}x_2 g(x_1) g(x_2) \hat{\sigma}_{gg \to A_0} , \tag{12}$$

where g is the gluon parton distribution function of the proton and x_i are the momentum fractions of the two gluons. By using the result of 2, show that

$$\sigma_{pp\to A_0} = \frac{\alpha_s^2}{16\pi}\frac{m_f^4}{v^2 m_A^2}\,|f(\tau)|^2\,\frac{1}{s}\int_{m_A^2/s}^{1}\frac{\mathrm{d}x_1}{x_1}g(x_1)g\left(\frac{m_A^2}{sx_1}\right)\,, \tag{13}$$

where $\alpha_s = g_s^2/(4\pi)$ and s is the total square momentum of the colliding protons.