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Exercise 1 [Tensors in Minkowski space]

Consider the following Poincaré transformation: $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + b^{\mu}$, with $\Lambda^{\mu}{}_{\nu}$ and b^{μ} constant.

Let $V^{\mu}(x^{\alpha})$ be a vector field in Minkowski space (i.e. it transforms as $V'^{\mu}(P') = \Lambda^{\mu}{}_{\nu} V^{\nu}(P)$, for any point P in spacetime), and ${x_{(i)}}^{\mu}(\lambda)$ the worldline of a particle.

How do the following quantities change under this transformation?

a) ${a_{(1)}}^{\mu} = {x_{(1)}}^{\mu}$

b) ${a_{(2)}}^{\mu} = \dfrac{\mathrm{d}{x_{(1)}}^{\mu}}{\mathrm{d}t}$

c) ${a_{(3)}}^{\mu} = \dfrac{d{x_{(1)}}^{\mu}}{d\tau}$

d) ${a_{(4)}}^{\mu} = {x_{(1)}}^{\mu} - {x_{(2)}}^{\mu}$

e) ${a_{(5)}}^{\mu}{}_{\nu} = \dfrac{\partial V^{\mu}}{\partial x^{\nu}}$

f) $\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & , (\mu,\nu,\rho,\sigma) \text{ is an even permutation of (0,1,2,3)} \\ -1 & , (\mu,\nu,\rho,\sigma) \text{ is an odd permutation of (0,1,2,3)} \\ 0 & , \text{otherwise} \end{cases}$

Which of the above quantities transform as Lorentz-tensors?

Hint: Write the determinant of a matrix $A^{\mu}{}_{\nu}$ using $\epsilon^{\mu\nu\rho\sigma}$.

Exercise 2 [Accelerated reference frames in Special Relativity I]

In the lecture, we have seen that the coordinate transformation between a coordinate system KS' (with coordinates x'^{μ}) which rotates with constant angular speed with respect to an inertial system IS (x^{α}) is

$$\begin{aligned} x &= x' \cos(\omega t') - y' \sin(\omega t') \\ y &= x' \sin(\omega t') + y' \cos(\omega t') \\ z &= z' \\ t &= t'. \end{aligned}$$

Show that the line element ds can be written as

$$\mathrm{d}s^2 = \left[c^2 - \omega^2(x'^2 + y'^2)\right] \mathrm{d}t'^2 + 2\omega y' \mathrm{d}x' \mathrm{d}t' - 2\omega x' \mathrm{d}y' \mathrm{d}t' - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2.$$

Exercise 3 [Accelerated reference frames in Special Relativity II]

Consider a spaceship that is accelerating with a constant acceleration a (in its rest frame) along the x-direction. Assume that it launches at $t = \tau = 0$ and determine:

(i) the proper time of the spaceship as a function of coordinate time $\tau(t)$ and vice versa.

(ii) coordinate distance as a function of proper time $x(\tau)$ and vice versa.

(iii) velocity as a function of coordinate and proper time $v(t)$ & $v(\tau)$. What happens in the $t \to \infty$ limit? Make a plot of $v(t)$.

After having done the calculations, we can start having fun:

(iv) Imagine you want to visit distant places with your spaceship that can constantly accelerate with $a = 1g$. Note that you should only accelerate for the first half of your journey. Use the second half to decelerate at the same rate. Calculate the proper time, coordinate time and maximal velocity for the following destinations: Mars (1.5 AU), Pluto (33 AU), Alpha Centauri (4.39 ly), Andromeda (2×10^6 ly) and the edge of the Universe (46.5×10^9 ly). Also, calculate the kinetic energy per unit mass at the instance when the spaceship has the highest velocity.

(v) Use the results above to explain the famous twin paradox: If a twin decides to go on a space mission while his brother stays on Earth, they will have aged differently when the space-twin returns. Make an example for such a space trip and the resulting age difference.

Hint: Express $u^\mu = dx^\mu/d\tau$ and $a^\mu = du^\mu/d\tau$ in the rest frame of the particle and in a general inertial frame. Recall that scalar products of 4-vectors are Lorentz-invariant. In particular, consider the products $u^\mu u_\mu$, $u^\mu a_\mu$ and $a^\mu a_\mu$. Remember that $v^\mu = dx^\mu/dt$.

Exercise 4 [Redshift in Special Relativity]

a) Imagine an observer A looking at another observer B located in A's reference frame along the x-axis at distance r, and travelling with a constant velocity $v = (\dot{x}, \dot{y}, 0)$.
B emits coherent light at A at some frequency ω (in its rest reference frame). What will the frequency of the light as observed by A be?

Hint: The frequency in a given reference frame is the time interval between two wavecrests in this frame. Consider this time interval to be small enough to be treated as infinitesimal.

The redshift z is defined as $1 + z = \dfrac{\omega_{\text{em}}}{\omega_{\text{obs}}}$. What are the limits of this quantity? Is it possible to have $z = 0$ for any given $|v|$? If so, what would the velocity be?

b) Imagine now two spaceships aligned on the x-axis, one at $x = 0$, and the other one at $x = R$, both feeling a constant acceleration $a = d^2x/dt^2$. At $t = 0$, both have velocity $v = 0$, and the first one emits a beam of light at frequency ω in the direction of the second. What is the redshift of the light as observed by the second spaceship? How large does the acceleration have to be for the redshift to be infinite? Why is that? How does that change when $d^2x/d\tau^2 = a$ is constant?