General Relativity
Sheet 11

HS 2019
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Course website: `http://www.physik.uzh.ch/en/teaching/PHY511`

Issued: 27.11.2019
Discussion: 05.12.2019

Exercise 1 [Computer-aided GR]

In your favourite computer algebra system (Mathematica, Maple, Sage, etc.), write a program to calculate the Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar, Einstein tensor, and geodesic equations for a given metric and coordinates. Use this to calculate the curvature invariant $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ of the Kerr metric

$$ds^2 = \left(1 - \frac{r_s r}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + \alpha^2 + \frac{r_s r \alpha^2}{\rho^2}\sin^2\theta\right)\sin^2\theta d\phi^2 + \frac{2 r_s r \alpha \sin^2\theta}{\rho^2} dt d\phi\,,$$

with

$$r_s = 2GM\,,$$
$$\rho^2 = r^2 + \alpha^2\cos^2\theta\,,$$
$$\Delta = r^2 - r_s r + \alpha^2\,.$$

It describes the vacuum solution outside a rotating black hole with spin parameter $\alpha = J/M$ and reduces to the Schwarzschild metric when $\alpha \to 0$. You can check your code by confirming that the Ricci tensor vanishes and that the curvature invariant reduces to the Schwarzschild value $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{48G^2M^2}{r^6}$ when $\alpha = 0$.

Exercise 2 [The maximally extended Schwarzschild solution]

Let us explore the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,.$$

One way to better understand the geometry of this spacetime is to consider the casual structure defined by light cones. Consider radial null geodesics ($ds^2 = 0$), meaning photons radially falling into a black hole or escaping from it. The slope of the light cone in a t-r diagram is then given by

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}\,.$$

From this we see that the slope goes to $\pm\infty$ as we approach the Schwarzschild radius. Thus a light ray which approaches $r = 2GM$ never seems to get there. An outside observer would never see anything fall into the black hole.
This effect is an illusion though and just a consequence of our coordinate system. In this exercise we want to look at different coordinate systems better suited to deal with the event horizon of a black hole.

a) Define a new radial coordinate r^* via

$$dr^* = \left(1 - \frac{2GM}{r}\right)^{-1} dr\,.$$

Write out the Schwarzschild metric with this new coordinate. You don't have to give an explicit expression for r in terms of r^*, just give an implicit equation and give the new metric in terms of $r(r^*)$. Show that radial null geodesics are given by the equation $t = \pm r^* + \text{const}$, but that now the Schwarzschild radius $r = 2GM$ is located infinitely far away.

b) Introduce new coordinates adapted to the null geodesics

$$v = t + r^* \,, \qquad u = t - r^* \,.$$

Now infalling radial null geodesics are given by $v = \text{const}$, while outgoing ones are given by $u = \text{const}$. Go back to the original coordinate r and write out the metric in either v-r or u-r coordinates. What can you say about the light cones in this coordinate system. Where is the Schwarzschild radius located?

c) Write the metric in the v-u coordinates and give an implicit equation for the original Schwarzschild coordinate r in terms of v and u. We see that the Schwarzschild radius is again located at infinity. To account for this, we pull these points to a finite coordinate value by introducing new coordinates

$$v' = e^{v/4GM} \,, \qquad u' = -e^{-u/4GM} \,.$$

Write out the metric in these coordinates.

d) Both v' and u' are null coordinates (The partial derivatives $\partial_{v'}$ and $\partial_{u'}$ are null vectors). We'd like to work with one timelike and three spacelike coordinates, so introduce new coordinates

$$T = \frac{1}{2}(v' + u') \,, \qquad R = \frac{1}{2}(v' - u') \,. \tag{1}$$

The coordinates (T, R, θ, ϕ) are known as **Kruskal coordinates**. Write the metric in these coordinates and give an implicit relation between T, R, r and t. What are the allowed values for T and R. What do the radial null geodesics look like? What happens at the Schwarzschild radius $r = 2GM$ and the real singularity $r = 0$? Describe all the different regions of the spacetime we encounter here.