

## 3. Fluid friction in liquids

### 3.1 Introduction

Generally the term *fluid* is understood to be matter either in the gaseous or liquid state. The physics involved on the macroscopic scale is essentially the same; the difference is orders of magnitude in density and therefore molecular distance and the corresponding forces acting. The experimental techniques required for investigating gases or liquid are quite different, though. In this text *fluid* refers to the liquid state, if not otherwise explicitly stated.

A fluid flows, e.g. in a tube, not as solid 'plug', but with more or less complicated internal motion with continuously changing velocities at all points of reference in the fluid. Resistance to flow stems from attractive forces between the molecules, and therefore work is required if molecules are to be separated, as in a flowing liquid. For the many molecules in a macroscopic fluid sample, it would be impractical to try to calculate this work on the molecular level. Instead, at least for engineering purposes, a different approach is adopted. As is often the case when the underlying physics is too complicated for direct access, or the effort would not be rewarding, a phenomenological view is applied, and the physical mechanisms described by using easily measured macroscopic material parameters. One such case is friction of any kind, and in particular the internal forces that make a fluid resist motion : the observed phenomenon is reduced to a parameter that we will examine below: *viscosity*; the collective resistance to motion by the molecular forces is referred to as *fluid friction*.

Returning briefly to the molecular level, we recall that polyatomic molecules are stabilized by internal, or *intramolecular* forces between atoms or groups of atoms, which are not of concern here, whilst fluid friction is determined by the forces between the molecules, the *intermolecular* forces.

We also note that the intermolecular forces are attractive, but in a fluid rather weak. The weak binding between molecules may typically be described as Van der Waals forces or hydrogen bonds, the latter, e.g. for the important case of water and watery solutions.

Practical and economical consequences of engineering design are found in all kind of installations involving extensive tubing, as is the case for heat exchangers, thermal (steam) power stations, nuclear power reactors, hydropower stations, oil refineries, chemical processing plants, paper mills, but also on a different scale, the system of human blood vessels; In all these cases careful design was employed to balance tubing diameter and viscosity against desired flow characteristics.

In this laboratory we investigate fluid friction in two important cases: the flow resistance of liquids

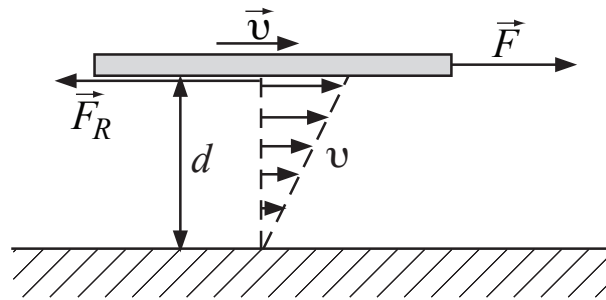


Figure 3.1: Principle of measuring the Newtonian fluid friction.

in capillary tubes, and its dependence on tube diameter, for water and castor oil (ricinus); In a second experiment, the resistance to motion of spherical balls falling by gravity, again in water and castor oil.

Both methods are developed for quantitative determination of fluid friction, and determine quantitatively the coefficient of viscosity

In particular we also introduce the *Reynolds number* as an important characteristic of fluid flow.

## 3.2 Theory

### a) Newtons law for Fluids – Newtonian fluids

A Newtonian fluid (named so for Isaac Newton) is a fluid whose stress versus strain rate curve is linear and passes through the origin. The constant of proportionality is known as the viscosity of the fluid<sup>1</sup>.

In common terms, this means that the fluid continues to flow, regardless of the forces acting on it. For example, water is Newtonian, because it continues to show fluid properties no matter how fast it is stirred or mixed. In contrast, a non-Newtonian fluid changes its properties: stirring can leave a 'hole' behind because the fluid becomes thicker (Rheopecty; this behaviour is seen in materials such as pudding, starch in water). The non-Newtonian fluid may also become thinner with stirring, the drop in viscosity causing it to flow better (Thixotropy; this property has been put to good use in non-drip paints, which easily wet the brush because they flow easily, but become more viscous(thicker) when applied by brushstroke on a wall).

Next we need to define two common notions used to describe fluid motion: *laminar flow* sometimes known as streamline flow, occurs when a fluid flows in parallel layers. The appearance is smooth.

Flow that is not laminar is termed *turbulent*. It is characterised by whirls, or in more technical language by eddies and vortices and generally by chaotic, property changes. There is rapid variation of pressure and velocity in space and time. The appearance is rough.

Extended piping systems, as used for district heating or chemical industry, need to be designed for

<sup>1</sup>The material properties stress, strain and shear are introduced in laboratory TB.

laminar flow since this requires the least pumping power and causes less vibrations and material deterioration. For applications where heat exchangers and reaction vessels are involved, turbulent flow is essential for good heat transfer and mixing, since in this flow regime there is flow also transversely to the general flow direction.

Employing a simple model for laminar flow, we may assume infinitely thin parallel fluid sheets move with different velocities such that frictional forces  $\vec{F}_R$  between the sheets act to set up shear stress. We also characterise  $\vec{F}_R$  as *drag*. This model is depicted in Fig. 3.1 : A layer of fluid is enclosed between two plates. If this frictional force causes the substance between the plates to undergo *shear flow* (as opposed to just elastic shearing, as for a solid, until the shear stress balances the applied force), the substance is called a fluid.

We assume here that the lower plate is stationary, and that the upper moves with a relative velocity  $v$ . It is further assumed that the attraction between the plates and the fluid is such that next to each plate there is a fluid sheet that does not move relative to the plate. As a consequence the velocities for each sheet changes with distance, as shown in the figure. In particular we note already here, that in the figure the change in velocity is linear with distance between the plate. This is not trivial, and we return to this subject later. The principle shown may be directly implemented in an apparatus for measuring fluid resistance, for instance by using two parallel rotating plates.

Under steady state conditions, an external force  $F = F_R$  is required to maintain the velocity  $v$  of the upper plate. For velocities not too high, and a plate distance  $d$  small compared to the area  $A$  of the upper plate (in order to avoid edge effects), the force  $F = F_R$  is proportional to the area and velocity  $v$  of the upper plate and inversely proportional to the distance:

$$F_R = \eta \cdot A \cdot \frac{v}{d} \quad (3.1)$$

Here  $\eta$  is a property variable characterising the "thickness" or "stiffness" of the liquid. Written in this form the relation is valid only for the chosen geometry of parallel plates; for the same liquid flowing, e.g. in a tube, the ratio  $v/d$ , the rate of *shear deformation*, varies with distance.

For straight, parallel and uniform flow, as in this case, it was postulated by Newton, that the shear stress,  $\tau$ , between layers is proportional to the *velocity gradient* in the direction perpendicular to the fluid sheets. This is the *Newton's criterion*, and may taken as a definition of a Newtonian fluid. The relation Eq. 3.1 then may be written on the equivalent but more general form:

$$\tau \equiv \frac{F_R(z)}{A} = \eta \cdot \frac{dv}{dz} \quad (3.2)$$

The factor of proportionality is by definition the viscosity, here designated  $\eta$ . Many fluids, such as water and most gases, satisfy Newton's criterion, that their flow be described by Eq. 3.2. By definition, the viscosity depends only on temperature and pressure, but not on the forces acting upon it.

It is worth noting, that it is the strength of the velocity gradient  $\frac{dv}{dz}$  perpendicular to the flow direction that determines the drag, the resistance to flow. In our intuitive picture of laminar flow, this is because of the friction in the fluid sheets gliding past each other. In practice this property

determines the time for a certain fluid to flow through a given tube, or the pumping power required for a certain volume throughput.

The coefficient of viscosity  $\eta$ , as defined through the relations Eq. 3.1 and Eq. 3.2, is the most common variant, and is therefore often called just *viscosity* or *absolute viscosity*. More specific names are the *dynamic viscosity*, or the *Newtonian viscosity*. The reader is advised, that several differently defined coefficient of viscosity are found in the literature (see below).

Non-Newtonian fluids exhibit a more complex relationship between shear stress and velocity gradient than simple linearity as in Eq. 3.2. Examples include polymer solutions, molten polymers, blood ketchup, shampoo, suspensions of starch<sup>2</sup>, many solid suspensions and most highly viscous fluids. It is not the 'thickness' *per se* that makes a fluid non-Newtonian, but the complicated interaction between large molecules or chains of molecules. As already mention above, an indication demonstrating the nonlinearity, is the drop in viscosity seen when stirring non-drip paints,

The viscosity is a strongly temperature dependent parameter, as we know already from everyday experience.

### Some common viscosity coefficients

Several different viscosity coefficients are in use, depending on the method for applying stress and the nature of the fluid, i.e. the type of application targeted. Specialised methods of measurement, and corresponding units are in use for industrial applications, in particular petrochemical. For Newtonian fluids, dynamic viscosity and kinematic viscosity are common, and often confused. If in doubt, always note the unit. We list some of the most common coefficients for the sake of completeness here:

Viscosity coefficients for Newtonian fluids

- *Dynamic viscosity* (introduced above) determines the dynamics of an incompressible Newtonian fluid.
- *Kinematic viscosity* is the dynamic viscosity divided by the density for a Newtonian fluid.
- *Volume viscosity* (or bulk viscosity) determines the dynamics of a compressible Newtonian fluid.

Viscosity coefficients for non-Newtonian fluids

- *Shear viscosity* is the viscosity coefficient when the applied stress is a shear stress (valid for non-Newtonian fluids).
- *Extensional viscosity* is the viscosity coefficient when the applied stress is an extensional stress (valid for non-Newtonian fluids; widely used for characterising polymers).

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<sup>2</sup>Those of the readers having been exposed to a British education may be familiar with the substance *oobleck*, which gets its name from the Dr. Seuss book *Bartholomew and the Oobleck*, where a goeey green substance, oobleck, fell from the sky and wreaked havoc in the kingdom.

Shear viscosity and dynamic viscosity are the best known in each of the two groups: both defined as the ratio between the pressure exerted on the surface of a fluid, in the lateral or horizontal direction, to the change in velocity of the fluid as you move down in the fluid (this is what is referred to as a velocity gradient).

### Units and nomenclature

Dynamic viscosity: The symbol commonly used for dynamic viscosity by mechanical and chemical engineers is  $\mu$ , whereas  $\eta$  is commonly preferred by chemists and IUPAC<sup>3</sup>.

The SI unit of dynamic viscosity is the pascal-second (Pa·s), which in SI base units is expressed as  $\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1}$ . If a fluid with a viscosity of one Pa·s is put between two plates, and one plate is displaced horizontally, creating a shear stress of one pascal, it moves a distance equal to the thickness of the layer between the plates in one second.

The cgs physical unit for dynamic viscosity is the poise (P)<sup>4</sup>. It is more commonly expressed, particularly in ASTM<sup>5</sup> standards, as centipoise (cP). Water at 20 °C has a viscosity of 1.0020 cP.  $1 \text{ P} = 1 \text{ g}\cdot\text{cm}^{-1}\cdot\text{s}^{-1}$ . The relation between poise and pascal-seconds is:  $10 \text{ P} = 1 \text{ kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1} = 1 \text{ Pa}\cdot\text{s}$ ,  $1 \text{ cP} = 0.001 \text{ Pa}\cdot\text{s} = 1 \text{ mPa}\cdot\text{s}$ . The name 'poiseuille' (Pl) has been proposed for this unit, also after Jean Louis Marie Poiseuille but has not been accepted internationally. Care must be taken in not confusing these units where they might appear.

Kinematic viscosity: In many situations we are concerned with the ratio of the viscous force to the inertial force per unit volume,  $\rho \cdot g$ . For this purpose the kinematic viscosity  $\nu$  is defined as:

$$\nu = \frac{\eta}{\rho} \quad (3.3)$$

where, as before,  $\eta$  is the dynamic viscosity [Pa·s],  $\rho$  is the mass density [ $\text{kg}\cdot\text{m}^{-3}$ ], and  $\nu$  is the kinematic viscosity [ $\text{m}^2\cdot\text{s}^{-1}$ ]. The cgs unit for kinematic viscosity is the stokes (St), named after George Gabriel Stokes. It is sometimes expressed in terms of centistokes (cSt or ctsk). 1 stokes = 100 centistokes =  $1 \text{ cm}^2\cdot\text{s}^{-1} = 0.0001 \text{ m}^2\cdot\text{s}^{-1}$ . 1 centistokes =  $1 \text{ mm}^2\cdot\text{s}^{-1} = 10^{-6} \text{ m}^2\cdot\text{s}^{-1}$ . The kinematic viscosity is sometimes referred to as *diffusivity of momentum*, because it is comparable to, and has the same SI-dimension [ $\text{m}^2\cdot\text{s}^{-1}$ ] as diffusivity of heat and diffusivity of mass. It is used in dimensionless numbers for the comparison of the ratio of the diffusivities.

$$1 \text{ Poise} = 1 \frac{\text{g}}{\text{cm}\cdot\text{s}} = 0.1 \text{ Pa}\cdot\text{s} \quad (3.4)$$

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<sup>3</sup>The International Union for Pure and Applied Chemistry

<sup>4</sup>named after Jean Louis Marie Poiseuille, who formulated Poiseuille's law of viscous flow.

<sup>5</sup>ASTM International (ASTM), originally known as the American Society for Testing and Materials, is an international standards organisation that develops and publishes voluntary consensus technical standards for a wide range of materials, products, systems, and services.

### b) The Hagen-Poiseuille Equation

Using Newton's law of fluids, as discussed above, we now develop an expression for laminar flow in a cylindrical tube of length  $l$  and inner radius  $R$  (cf Fig. 3.2).

For the analysis we choose a concentric cylindrical volume element of fluid of radius  $r < R$ . Because of a pressure difference  $\Delta p = p_1 - p_2$  persisting between the ends of the tube, a net force  $F_p$  acts on the cylindrical element:

$$F_p = \pi \cdot r^2 \cdot \Delta p \quad (3.5)$$

According to Eq. 3.2, the velocity gradient perpendicular to the cylinder axis  $(dv/dr)|_r$  gives rise to a drag force  $F_R$  at the distance  $r$  from the axis:

$$F_R = \eta \cdot 2\pi \cdot r \cdot l \cdot \left( \frac{dv}{dr} \right) \Big|_r \quad (3.6)$$

For stationary conditions (no change in the velocity profile with time), a balance between the pressure induced force  $F_p$  and the frictional drag  $F_R$  persists, which we write as:

$$\pi \cdot r^2 \cdot \Delta p + \eta \cdot 2\pi \cdot r \cdot l \cdot \left( \frac{dv}{dr} \right) \Big|_r = 0 \quad (3.7)$$

From this we may express the velocity gradient at distance  $r$  from the cylinder axis

$$\left( \frac{dv}{dr} \right) \Big|_r = -\frac{\Delta p}{2\eta \cdot l} \cdot r \quad (3.8)$$

The actual flow speed  $v(r)$  may be obtained by solving this differential equation, which because of the simple form amounts to an integration using the given boundary condition  $v(r = R) = 0$ :

$$v(r) = \frac{\Delta p}{4\eta \cdot l} \cdot (R^2 - r^2) \quad (3.9)$$

This is a parabolic velocity distribution as shown in Fig. 3.3. The maximum velocity is found at  $r = 0$  in the centre of the tube, from where it decreases quadratically with increasing  $r$ , becoming zero at the wall of the tube.

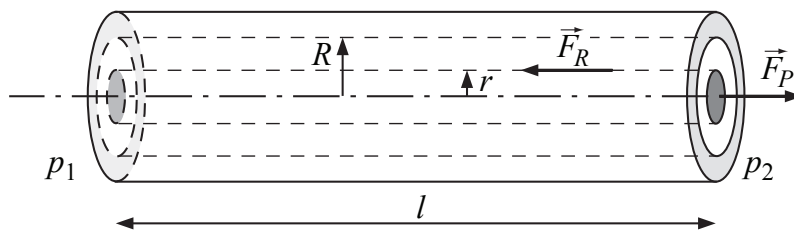


Figure 3.2: Forces and pressures acting on a fluid in a section of a cylindrical tube.

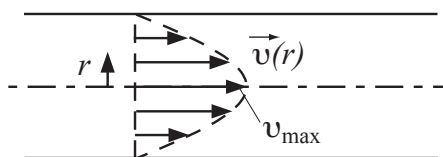


Figure 3.3: Velocity profile in a cylindrical tube.

Using the geometry of Fig. 3.4, the amount  $dQ$  of fluid flowing through an annular cylinder of radius  $r$  in time  $t$ ,) may be expressed as:

$$dQ = t \cdot v(r) \cdot 2\pi \cdot r \cdot dr = t \cdot \frac{\pi \cdot r \cdot \Delta p \cdot (R^2 - r^2)}{2\eta \cdot l} dr \quad (3.10)$$

Integrating this equation over the cross section of the tube we arrive at the Hagen-Poiseuille Equation for for the amount  $Q$  flowing in tie  $t$ :

$$Q = t \cdot \int_0^R \frac{\pi \cdot r \cdot \Delta p \cdot (R^2 - r^2)}{2\eta \cdot l} dr = \frac{\pi \cdot R^4 \cdot \Delta p}{8\eta \cdot l} \cdot t \quad (3.11)$$

The reader is urged to perform a dimensional analysis of this expression in order to determine the correct unit of  $Q$ .

If the dimensions of the tube is known, the viscosity  $\eta$  of the fluid given may be determined from measured values of  $Q$ ,  $t$  and  $\Delta p$ .

### c) Laminar and turbulent flow – Reynolds number.

Newtons equation 3.2 is valid only for the case of laminar flow, as discussed above. For flow that is faster than a certain critical value  $v_{krit}$  the laminar flow lines are disturbed by an unruly behaviour and may develop vortex patterns that depends on the given geometry. This is the regime of turbulent flow that is characterised by fluid motion also transversely to the direction of flow. The complicated flow pattern of turbulent flow requires more power for the same volume throughput, which is equivalent to a higher flow resistance. For a given system of fluid and geometry, it is found that the critical velocity  $v_{krit}$  is determined by the mass density  $\rho$  the viscosity  $\eta$  and a characteristic dimension  $d$  of the system (e.g. the diameter of the tube in case of a cylindrical tube). In the study of macroscopic phenomenology of physics and engineering, surprisingly simple dimensionless combinations of measurable entities have proved themselves to faithfully characterise otherwise highly complicated systems.

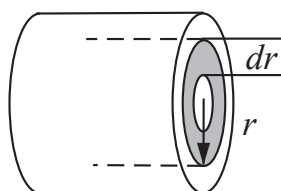


Figure 3.4: Geometry for the development of Hagen-Poiseuille Equation.

As a characteristic of fluid flow, the dimensionless number defined as

$$Re = \frac{\rho \cdot v \cdot d}{\eta} \quad (3.12)$$

has emerged through measurement and intuition. This is *Reynold's number*, and it turns out that below a certain value  $Re_{krit}$  laminar flow is present, whereas for  $Re > Re_{krit}$  turbulent flow prevails.

For straight, cylindrical tubes  $Re_{krit} \approx 2300$ . Eq. 3.12 the gives us:

$$v_{krit} = 2300 \frac{\eta}{\rho \cdot d} \quad (3.13)$$

where  $d$  is the diameter of the tube. The flow remains laminar as long as the mean flow velocity is below  $v_{krit}$ . The remarkable feature of Reynold's number, or any other of several dimensionless number, is that any combination of the values of the measurable properties, consistently describe properties of the material.

#### d) Stoke's equation

For highly viscous fluids (high  $\eta$ ), such as oil at low temperatures, the throughput in a capillary would be so slow that the viscosity  $\eta$  could not practically be determined. In such cases an effective and simple alternative method is to measure the time of fall by gravity of a sphere in the fluid of interest. The forces acting on the sphere is shown in Fig. 3.5. The force of gravity  $F_G$  and the bouyancy  $F_A$  are respectively:

$$F_G = m \cdot g = \frac{4}{3} \pi \cdot r^3 \rho_K \cdot g$$

$$F_A = \frac{4}{3} \pi \cdot r^3 \cdot \rho_{Fl} \cdot g$$

where  $\rho_K$  is the mass density and  $r$  the radius of the sphere and  $\rho_{Fl}$  the density of the fluid.

For the drag force  $F_R$ , we have for laminar flow the frictional law of Stoke:

$$F_R = 6 \pi \cdot \eta \cdot r \cdot v \quad (3.14)$$

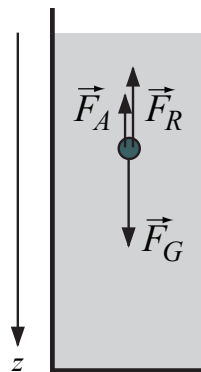


Figure 3.5: Forces acting on a sphere falling through a viscous fluid.



where

$$\begin{aligned}\eta &= \text{viscosity of the fluid} \\ r &= \text{radius of the fluid} \\ v &= \text{terminal velocity of the sphere}\end{aligned}$$

The frictional force  $F_R$  is proportional to the velocity and thus for some velocity, the *terminal velocity*, a stationary condition is reached with constant velocity (see appendix). The forces acting on the sphere balances each other according to the relation:

$$\frac{4}{3} \pi \cdot r^3 \cdot (\rho_K - \rho_{Fl}) \cdot g - 6 \pi \cdot \eta \cdot r \cdot v = 0 \quad (3.15)$$

If the mass density  $\rho$  and the radius of sphere  $R$  are known, the viscosity  $\eta$  may be determined from a measured value of the terminal velocity.

$$\eta = \frac{2 r^2 \cdot g \cdot (\rho_K - \rho_{Fl})}{9 v} \quad (3.16)$$

### 3.3 Experimental

#### a) Determination of the viscosity of water

The viscosity of water is determined according to Eq. 3.11. The amount of water  $Q$  collected from a capillary of radius  $R$  and length  $l$  in time  $\Delta t$  is determined<sup>6</sup>?. The measurement is made using three different capillaries (five are provided, marked with roman numerals I - V). Check which capillaries that work – the two smallest diameters probably does not allow laminar flow, and should therefore not be used (recall that the equations developed are valid only for laminar flow). Make sure that de-mineralized (or distilled) water is used in the experiment, since otherwise deposits are left in the capillaries (de-mineralized water may be found in rooms 11 G 24 and 11 G 26: water taps with green rings and text). With most capillaries there is some dripping at the same time as the water flows nicely. It is left to the discretion of the experimentalist to handle this matter in a reasonable way.

The experimental set-up is shown in Fig. 3.6. A vertical glass cylinder with volume markings has a provision for mounting the capillaries at the output in the lower part of the cylinder. For given height  $h$  of the fluid column, the pressure difference  $\Delta p = p_1 - p_2$  between the ends of the capillaries will be:

$$\Delta p = p_1 - p_2 = p_L + \rho \cdot g \cdot h - p_L = \rho \cdot g \cdot h \quad (3.17)$$

- Measure the length  $l$  of the capillaries. The inner diameter is given at the lab desk. Mount one of the capillaries at the outlet of the cylinder.
- Fill the cylinder with de-mineralised water. For each of the different capillaries there might be a highest level above which the flow through the capillary will not be laminar. This has to be tested. The same might be true for a lowest level.

<sup>6</sup>Make certain that you know the meaning of  $Q$ : what is the SI-unit in 3.11

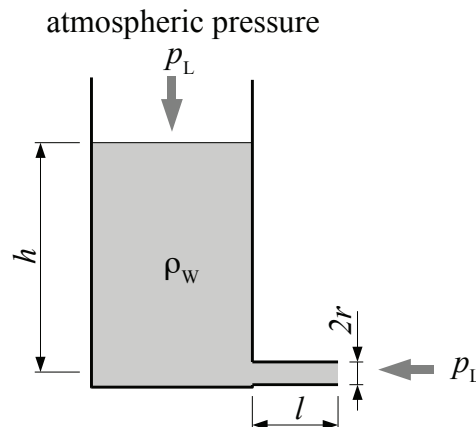


Figure 3.6: Set-up for determining the viscosity of water.

- Let the water running from the capillary collect in a cup during the time interval  $t$ , for known heights  $h_a$  and  $h_b$ . The time should not be too short, since then errors might become large.
- Determine the amount of water  $Q$  using the electronic scale, and correct for the mass of the empty cup.
- Calculate the viscosity  $\eta$  according to Eq. 3.11. Use for  $\Delta p$  the average of the pressure differences at the end and at the beginning of the experiment <sup>7</sup>.

$$\Delta p = \frac{\Delta p_a + \Delta p_e}{2} = \rho_W \cdot g \cdot \frac{h_a + h_e}{2} \quad (3.18)$$

- Repeat the measurement for the other two capillaries. Calculate the average of the values for the viscosity  $\eta$ , and calculate the errors according to standard procedures. If the method is sound, the value for  $\eta$  should not depend on the capillary. Comment upon this in your report.
- For all three capillaries calculate the mean flow velocity, i.e. assume for this calculation that the fluid is moving through the tube as a solid 'plug'. Verify that this velocity is below the critical velocity for laminar flow.

### b) Determination of viscosity for castor oil

In this case of a viscous fluid, that certainly would not flow through our capillaries, we determine the viscosity using Eq. 3.16. Spherical steel balls fall by gravity in a tube filled with castor oil, the time for the balls falling a certain distance is registered. Balls with three different radii are used in consecutive experiments. Since the viscosity depends strongly on temperature, the temperature of the oil will have to be measured continuously.

- As a preparation, perform the experiment for a mid-sized steel ball, and make qualitative observation.
- From this experience, choose a suitable vertical distance over which the falling balls will be monitored. Mark off the beginning and the end of the distance on the cylinder and measure the height  $\Delta l$ . Make certain that the balls have reached the terminal velocity before you mark off the starting point. The distance should be chosen so that it can be conveniently stopped with a manual timer, without unnecessary large errors.
- Choose another ball of the same size and measure the diameter with a micrometer. Measure the time  $\Delta t$  for the balls to fall through the distance chosen. Calculate the velocity  $v = \Delta l / \Delta t$  and the viscosity  $\eta$  according to Eq. 3.16. Use the radius measured and the following values for the densities:

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<sup>7</sup>A certain height  $h$  of water column corresponds to a pressure difference between the pressure at height  $h$  and the pressure at the inlet of the capillary, or any other chosen reference point. It is therefore irrelevant from where  $h$  is measured.

- Castor oil:  $\rho_{Fl} = 0.96 \times 10^3 \text{ kg m}^{-3}$
- Steel ball:  $\rho_K = 7.86 \times 10^3 \text{ kg m}^{-3}$
  
- Repeat the measurement for five different mid-sized balls, and then for five different balls of larger, and five different of smaller size. If necessary, chose different height distances for balls of different size.
  
- Determine the average of the results and calculate the errors appropriately.

### 3.4 Appendix

The equation of motion for the falling ball (Newton's law of motion) is developed from the generic form:

$$F = m \cdot a \quad (3.19)$$

For the acceleration we write:

$$a \equiv \frac{d^2}{dt^2} \equiv \frac{dv}{dt} \quad (3.20)$$

With notation and reference direction as in Fig. 3.5, the external forces acting are:

$$F = F_G - F_A - F_R \quad (3.21)$$

In particular we know that  $F_R = 6\pi \cdot \eta \cdot r \cdot v$ , which we for simplicity write as  $F_R = \beta \cdot v$ . For the same reason we write  $\alpha = F_G - F_A$ , from which we then have the equation of motion as a first order ordinary differential equation (ODE) in  $v = dz/dt$ :

$$m \cdot \frac{dv}{dt} = \alpha - \beta \cdot v \quad (3.22)$$

Note, that  $\alpha$  is always zero, if we may assume that the density of the ball is greater than that of the fluid. The solution to the differential equation could be easily found with conventional methods (it is an exponential determined by the boundary conditions), we chose here to discuss the relation phenomenologically. Although we are interested only in the stationary solution, we start with a general approach to this equation.

Positive reference direction is in the direction of gravitation. Assuming that the initial speed of the ball is zero, then, since  $\alpha > 0$ , the left side of the equation also has to be positive, i.e.  $dv/dt > 0$  meaning that the speed increases (in a more direct view of matters, this has to be the case since the ball falls by gravity because of the higher density of the ball) Since  $\alpha$  is independent of the velocity, and the term  $\beta \cdot v$  now increases, but has a negative sign, the right side of the ODE will decrease, requiring the derivative on the left side to become smaller. Eventually a stationary condition is reached as the ball attains its (constant) terminal velocity.

In the limit  $t \rightarrow \infty$  we have  $dv/dt = 0$  and  $v_\infty = \alpha/\beta$ , and the solution to the ODE:

$$v(t) = \frac{\alpha}{\beta} \cdot \left(1 - e^{-\frac{\beta}{m}t}\right) \quad (3.23)$$

The higher the viscosity, the faster the terminal velocity is reached. If the velocity of the ball has some finite value, either larger or smaller than the terminal velocity, an equivalent reasoning may be applied, leading in any case to an asymptotical approach of the terminal velocity  $v_\infty = \alpha/\beta$  as shown in Fig. 3.7 for three different examples of  $v(t = 0)$ .

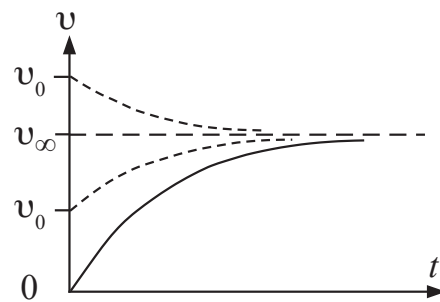


Figure 3.7: Change of velocity for a ball falling in a viscous fluid for three different initial velocities:  $v(t = 0) = 0$  (solid line) and two different  $v(t = 0) > 0$  (broken lines).