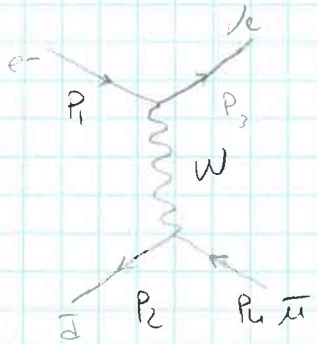
KT II - EXERCISE SHEET 6

ELECTRON - QUARK SCATTERING

a) $e^- q \rightarrow \nu_e q$



$$-i\,\mathcal{M}_{fi} = \left[-i\frac{g_W}{\sqrt{2}}\,\bar{u}(3)\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,u(1)\right]\frac{i\,g_{\mu\nu}}{m_W^2}\left[-i\frac{g_W}{\sqrt{2}}\,\bar{v}(2)\,\gamma^\nu\frac{1}{2}(1-\gamma^5)\,v(4)\right]$$

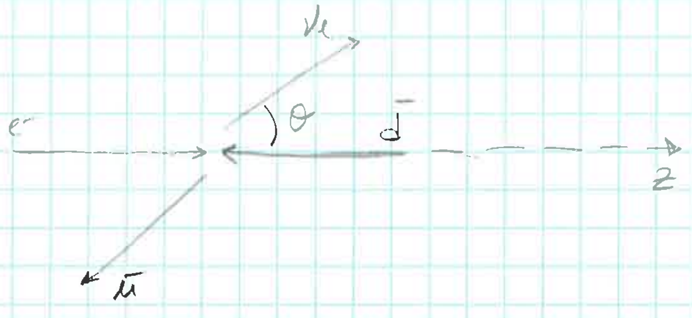
$$\mathcal{M}_{fi} = \frac{g_W^2}{2m_W^2}\,g_{\mu\nu}\left[\bar{u}(3)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(1)\right]\left[\bar{v}(2)\,\gamma^\nu\frac{1}{2}(1-\gamma^5)\,v(4)\right]$$

$$= \frac{g_W^2}{2m_W^2}\; j_e \cdot j_q$$

where $j_e = \left[\bar{u}_\downarrow(3)\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,u_\downarrow(1)\right]$ and $j_q = \left[\bar{v}_\uparrow(2)\,\gamma^\nu\frac{1}{2}(1-\gamma^5)\,v_\uparrow(4)\right]$

In high energy limit both lepton and quark masses can be neglected and LH chiral states are effectively identical to LH helicity states (same for RH states)

So, in this limit only LH helicity particles and RH helicity anti-particles participate in the weak interaction



$(\theta_1, \phi_1) = (0, 0)$

$(\theta_2, \phi_2) = (\pi, \pi)$

$(\theta_3, \phi_3) = (\theta^*, 0)$

$(\theta_4, \phi_4) = (\pi - \theta^*, \pi)$

The corresponding spinors are:

$$u_\downarrow(1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \qquad u_\downarrow(3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix}$$

$$v_1(2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \qquad v_1(4) = \sqrt{E}\begin{pmatrix}\sin\left(\frac{\pi-\theta^*}{2}\right)\\+\cos\left(\frac{\pi-\theta^*}{2}\right)\\-\sin\left(\frac{\pi-\theta^*}{2}\right)\\-\cos\left(\frac{\pi-\theta^*}{2}\right)\end{pmatrix} = \sqrt{E}\begin{pmatrix}\cos\frac{\theta^*}{2}\\\sin\frac{\theta^*}{2}\\-\cos\frac{\theta^*}{2}\\-\sin\frac{\theta^*}{2}\end{pmatrix} = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix}$$

So $$j_e^\mu = E\,(-s\ \ c\ \ s\ \ -c)\,\gamma^0\gamma^\mu\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

$$j_e^0 = 2Ec$$

$$j_e^i = E\,(-s\ c\ s\ -c)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} =$$

$$= E\,(-s\ c\ s\ -c)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\sigma_{12}^i\\\sigma_{22}^i\\\sigma_{12}^i\\\sigma_{22}^i\end{pmatrix} =$$

$$= E\,(-s\ c\ s\ -c)\begin{pmatrix}\sigma_{12}^i\\\sigma_{22}^i\\-\sigma_{12}^i\\-\sigma_{22}^i\end{pmatrix} = E\,(-s\sigma_{12}^i + c\sigma_{22}^i - s\sigma_{12}^i + c\sigma_{22}^i)$$

$$= -2E\,(s\sigma_{12}^i - c\sigma_{22}^i)$$

$$\Rightarrow\ j_e^1 = -2Es \qquad \sigma^1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$j_e^2 = +2iEs \qquad \sigma^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$j_e^3 = +2Ec \qquad \sigma^3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

- $j_e^\mu = 2E\,(c,\ -s,\ is,\ c)$

$$j_q^\nu = E\,(1\ \ 0\ \ -1\ \ 0)\,\gamma^0\gamma^\nu\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix}$$

$$j_q^0 = E\,(c+c) = 2Ec$$

$$j_q^i = E(1\ 0\ -1\ 0)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix} =$$

$$= E(1\ 0\ -1\ 0)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}-c\sigma_{11}^i - s\sigma_{12}^i\\-c\sigma_{21}^i - s\sigma_{22}^i\\-c\sigma_{11}^i - s\sigma_{12}^i\\-c\sigma_{21}^i - s\sigma_{22}^i\end{pmatrix} =$$

$$= E(1\ 0\ -1\ 0)\begin{pmatrix}-c\sigma_{11}^i - s\sigma_{12}^i\\-c\sigma_{21}^i - s\sigma_{22}^i\\c\sigma_{11}^i + s\sigma_{12}^i\\c\sigma_{21}^i + s\sigma_{22}^i\end{pmatrix} = E\left(-c\sigma_{11}^i - s\sigma_{12}^i - c\sigma_{11}^i - s\sigma_{12}^i\right) =$$

$$= -2E\left(c\sigma_{11}^i + s\sigma_{12}^i\right)$$

$$j_q^1 = -2E\,s \qquad\qquad j_q^\nu = 2E\,(c, -s, is, -c)$$

$$j_q^2 = +2E\,is$$

$$j_q^3 = -2E\,c$$

$$M_{fi} = \frac{g_w^2}{2m_w^2}\,4E^2\,(c, -s, is, c)\,(c, -s, is, -c)$$

$$= \frac{g_w^2}{2m_w^2}\,4E^2\left(c^2 - s^2 + s^2 + c^2\right) = \frac{g_w^2}{m_w^2}\,\underset{\hat{s}}{4E^2}\,c^2 = \frac{g_w^2}{m_w^2}\,\hat{s}\,c^2$$

$\hat{s}$ → ant? of mean energy

$$M_{fi} = \frac{g_w^2}{m_w^2}\,\hat{s}\cdot\left(\frac{1+\cos\theta^*}{2}\right)$$

$$\frac{G_F}{\sqrt{2}} = \frac{g_w^2}{8m_w^2}$$

$$\langle |M_{fi}|^2\rangle = \frac{1}{4}\left(\frac{g_w^2}{m_w^2}\right)^2 \hat{s}^2\left(\frac{1+\cos\theta^*}{2}\right)^2$$

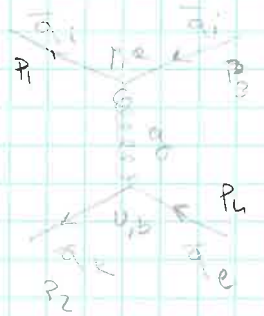
$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2\hat{s}}\,\langle|M_{fi}|^2\rangle = \frac{1}{64\pi^2\hat{s}}\,\frac{1}{4}\,\frac{1}{4}\left(\frac{g_w^2}{m_w^2}\right)^2\hat{s}^2\,(1+\cos^*\theta)^2 \, .$$

$$= \frac{64\,G_F^2}{64\cdot 2\cdot 16\pi^2}\,(1+\cos^*\theta)^2\,\hat{s} = \frac{G_F^2}{32\pi^2}\,(1+\cos\theta)^2\,\hat{s}$$

$$\int (1+\cos\theta^{+})^2 \, d\Omega^{+} = \int_0^{2\pi} d\phi^{+} \int_{-1}^{+1} (1 + \cos\theta^{+})^2 \, d\cos\theta = \frac{16\pi}{3}$$

$$\sigma_{e\bar{q}} = \frac{G_F^2}{6\pi} \hat{s}$$

COLOUR FACTORS



$$-iM_{d_1} = \bar v(1)\left\{-\frac{1}{2} i g_s \lambda^a \gamma^\mu\right\} v(3)$$

$$-iM_{d_1} = \left[\bar v(1)\left\{-\frac{1}{2} i g_s \lambda^a_{ij} \gamma^\mu\right\} v(2)\right] \frac{-i g_{\mu\nu}}{q^2}\delta^{ab}\left[\bar v(2)\left\{-\frac{1}{2} i g_s \lambda^b_{ke}\gamma^\nu\right\} v(4)\right]$$

$$M = -\frac{g_s^2}{4}\lambda^a_{ij}\lambda^a_{ke}\frac{1}{q^2} g_{\mu\nu}\left[\bar v(1)\gamma^\mu v(3)\right]\left[\bar v(2)\gamma^\nu v(4)\right]$$

due to δ^{ab}

$\lambda^a_{ij}\lambda^a_{ke} \to$ the order of indices is swapped wr to the quark case $ij \to ji$

$$C_{ik\to je} \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{ke} \Rightarrow \text{Colour factor}$$

$\to$ the index associated to the adjoint appear first

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\quad \lambda_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\quad \lambda_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \lambda_8 = \frac{1}{\sqrt3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

N.B. $\quad c_1 = r = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad c_2 = g\begin{pmatrix}0\\1\\0\end{pmatrix} \quad c_3 = b = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\lambda^a_{ji} = c_j^+\lambda^a c_i = c_j^+\begin{pmatrix}\lambda^a_{1i}\\\lambda^a_{2i}\\\lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

For antiparticles $\quad \lambda^a_{ij} = c_i^+\lambda^a c_j = c_i^+\begin{pmatrix}\lambda^a_{1j}\\\lambda^a_{2j}\\\lambda^a_{3j}\end{pmatrix} = \lambda^a_{ij}$

Possible cases $\Rightarrow$
- initial and final state part. have same colour : $\bar r\bar r \to \bar r\bar r$
- colour of init. particle is not the same but final part have the same colour of init part : $\bar r\bar b \to \bar r\bar b$
- 2 initial part have diff. colour and exchange colour : $\bar r\bar b \to \bar b\bar r$
- final state have different colour with respect to initial state : $\bar r\bar b \to \bar b\bar g$

$\Rightarrow$ Colour factor in this case is zero because colour is conserved.

$$C(\bar i k \to \bar j l) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ij}\lambda^a_{kl}$$ sum over types of gluons

$$\Rightarrow C(\bar r\bar r \to \bar r\bar r) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{11} = \frac{1}{4}\left(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11}\right) = \frac{1}{3}$$

$$C(\bar b\bar b \to \bar b\bar b) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{33}\lambda^a_{33} = \frac{1}{4}\left(\lambda^8_{33}\lambda^8_{33}\right) = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}$$

$$C(\bar g\bar g \to \bar g\bar g) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{22}\lambda^a_{22} = \frac{1}{4}\left(\lambda^3_{22}\lambda^3_{22} + \lambda^8_{22}\lambda^8_{22}\right) =$$

$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

$$\Rightarrow C(\bar r\bar b \to \bar r\bar b) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{33} = \frac{1}{4}\lambda^8_{11}\lambda^8_{33} = -\frac{1}{6}$$

$$= C(\bar r\bar g \to \bar r\bar g) = C(\bar b\bar r \to \bar b\bar r) = C(\bar g\bar r \to \bar g\bar r) =$$

$$= C(\bar g\bar b \to \bar g\bar b) = C(\bar b\bar g \to \bar b\bar g)$$

$$\Rightarrow C(\bar r\bar b \to \bar b\bar r) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{13}\lambda^a_{31} = \frac{1}{4}\left(\lambda^4_{13}\lambda^4_{31} + \lambda^5_{13}\lambda^5_{31}\right) =$$

Note: indexes are swapped because we are dealing with antiquarks.

$$= \frac{1}{4}(1+1) = \frac{1}{2}$$

$$= C(\bar r\bar g \to \bar g\bar r) = C(\bar b\bar r \to \bar r\bar b) = C(\bar g\bar r \to \bar r\bar g) =$$

$$= C(\bar g\bar b \to \bar b\bar g) = C(\bar b\bar g \to \bar g\bar b)$$

We have considered a particular colour exchange $i\bar{k} \to j\bar{l}$ but how many possible initial-state colour combinations do we have? 9

$$\langle |M|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |M(i\bar{k} \to j\bar{l})|^2$$

$$\Downarrow$$

$$\langle |C|^2 \rangle = \frac{1}{9} \sum |C(i\bar{k} \to j\bar{l})|^2$$

$\Rightarrow$ We can evaluate this from the single colour factor computed before

$$\langle |C|^2 \rangle = \frac{1}{9} \Big[3 \times \left(\frac{1}{3}\right)^2 + 6 \times \left(-\frac{1}{6}\right)^2 + 6 \times \left(\frac{1}{2}\right)^2 \Big] = \frac{2}{9}$$

3 of type $c\bar{c} \to c\bar{c}$

6 of type $c\bar{c}' \to c\bar{c}'$

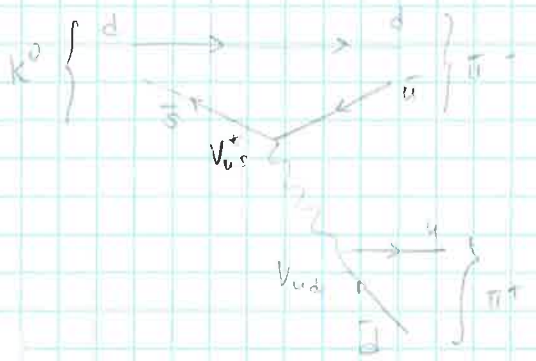
6 of type $c\bar{c}' \to c'\bar{c}$

KAON DECAY

$\bar{K}^0/K^0\ (d\bar{s})$ $\quad \pi^+ (u\bar{d})$ $\ \pi^-(\bar{u}d)$ $\ \pi^0 (u\bar{u}/d\bar{d})$

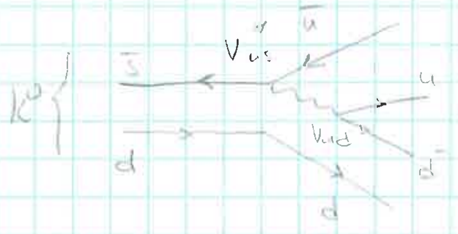
$K^0 \to \pi^+ \pi^-$

$d\bar{s} \to u\bar{d}\ \ \bar{u}d$



$K^0 \to \pi^0 \pi^0$

$d\bar{s} \to u\bar{u}\ \ d\bar{d}$



$M \propto |V_{us}|\,|V_{ud}| \approx \sin\theta_c \cos\theta_c$

D^0 DECAY

$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^+)} \approx 4\times 10^{-3}$$

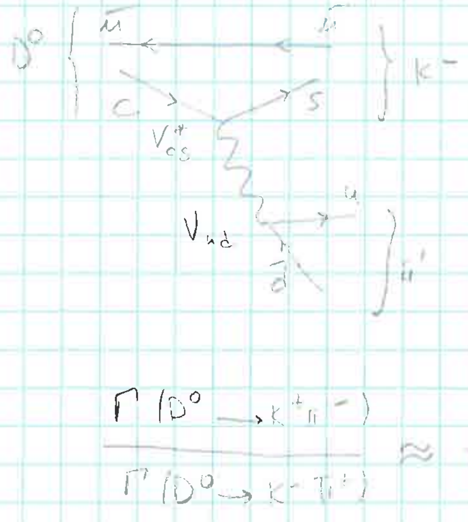
$D^0\ (c\bar{u})$

$K^+\ (u\bar{s})$

$K^-\ (\bar{u}s)$

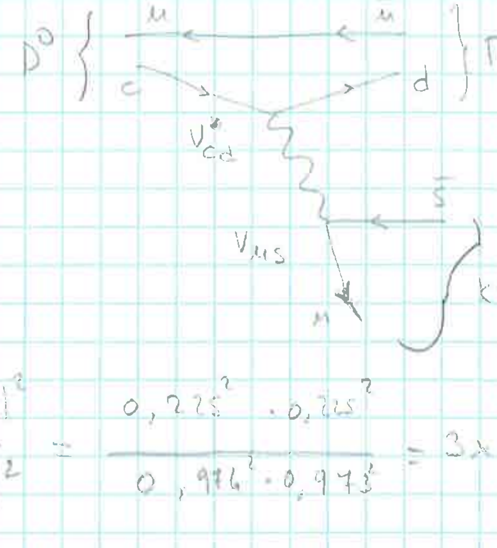
$D^0 \to K^-\pi^+$

$c\bar{u} \to \bar{u}s\ \ u\bar{d}$



$D^0 \to K^+\pi^-$

$c\bar{u} \to u\bar{s}\ \ \bar{u}d$



$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^+)} \approx \frac{|V_{cd}|^2\,|V_{us}|^2}{|V_{ud}|^2\,|V_{cs}|^2} = \frac{0{,}225^2 \cdot 0{,}225^2}{0{,}976^2 \cdot 0{,}973^2} = 3\times 10^{-3}$$

PARTICLE DECAYS

$\Lambda^0 \longrightarrow p + \pi^+$ $\Rightarrow$ charge is not conserved

$B^0 \longrightarrow D^- + \pi^+$ $\Rightarrow$

$(d\bar{b}) \longrightarrow (\bar{c}d) + (u\bar{d})$ $\Rightarrow$ Weak interaction $(b \to c)$

$\Lambda^0 \longrightarrow K^- + \pi^+$ $\Rightarrow$ violates Baryon conservation

$(uds) \longrightarrow (\bar{u}s)$ (u d)

KAON DECAY