MMP 2
Solutions 10

FS 19
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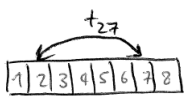
Exercise 1

Permutations are series of transposisions:



$$(t_{kl})^{-1} = t_{kl}$$

$$\pi_i = \prod_{j=1}^{N} t_{k_j l_j}, \qquad \mathrm{sgn}(\pi_i) = \begin{cases} -1, & \text{odd numbers of traspositions} \\ 1, & \text{even number} \end{cases}$$

a)

$$\rho(\pi_1)(\pi_2) = \underbrace{\mathrm{sgn}(\pi_1)}_{n \text{ transpositions}} \overbrace{\mathrm{sgn}(\pi_2)}^{m \text{ transpositions}} = \underbrace{\mathrm{sgn}(\pi_1\pi_2)}_{n\,+\,\mathrm{m}\text{ transpositions}} = \underline{\rho(\pi_1\pi_2)}$$

$$A_n = \{\pi \in S_n \quad | \ \mathrm{sng}(\pi) = 1\} \quad \text{invariant subgroup?}$$

i) **<u>Subgroup:</u>**

1) <u>neutral element e:</u>

$$\begin{aligned} \text{no transpositions} \quad &\Rightarrow \quad \mathrm{sgn}(e) = 1 = \rho(e) \\ &\Rightarrow \quad \underline{e \in A_n} \qquad \surd \end{aligned}$$

2) <u>inverse element g^{-1}</u>:

$$\rho(g^{-1}) \underset{\text{lecture}}{=} \rho(g)^{-1} \overset{(*)}{=} \frac{1}{\mathrm{sgn}(g)} = \mathrm{sgn}(g) = 1$$

$$\Rightarrow \forall g \in A_n, \quad \forall g^{-1} \in A_n \qquad \surd$$

$$(*): \quad \boxed{\mathrm{sgn}(gg^{-1}) = \mathrm{sgn}(g)\mathrm{sgn}(g^{-1}) = 1}$$

3) <u>closure:</u>

$$\rho(g_1) = \mathrm{sgn}(g_1) = 1, \qquad \rho(g_2) = \mathrm{sgn}(g_2) = 1$$
$$\Rightarrow \rho(g_1 g_2) = \mathrm{sgn}(g_1) \cdot \mathrm{sgn}(g_2) = 1$$
$$\Rightarrow \quad \underline{\forall g_1 g_2 \in A_n, \quad g_1 g_2 \in A_n}$$

ii) **<u>Invariance:</u>**

$$\pi_1 \in A_n, \qquad \pi_2 \in S_n$$

$$\Rightarrow \quad \text{invariant subgroup} \quad \Leftrightarrow \quad \pi_2^{-1}\pi_1\pi_2 \in A_n \forall \pi_2 \in S_n$$

$$\underset{\text{representation}}{\Longrightarrow} \rho(\pi_2^{-1})\rho(\pi_1)\rho(\pi_2) = \mathrm{sgn}(\pi_2^{-1})\overbrace{\mathrm{sgn}(\pi_1)}^{=1} \cdot \mathrm{sgn}(\pi_2)$$
$$= \mathrm{sgn}(\pi_1^{-1})\mathrm{sgn}(\pi_2) = 1 \quad \forall \pi_2 \in S_n$$

$$\Rightarrow \quad \underline{A_n \text{ is invariant under conjugation}}$$

$\square$

b) Since $t_{kl} = (t_{kl})^{-1} = t_{lk}$:

$$\rho(\underbrace{t_{kl}t_{kl}^{-1}}_{e}) = \rho(t_{kl}^2) = \underline{\rho(t_{kl})^2 = 1}$$

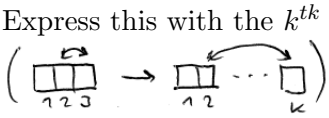
$\Rightarrow \quad \rho(t_{kl})$is either 1 or -1 ($t_{kl}$ is just one <u>swap</u>)

Use the identity: $\boxed{t_{12}t_{23}t_{12} = t_{23}t_{12}t_{23}}$

Check it first:

$$\begin{aligned}
\underline{t_{12}t_{23}t_{12}(A,B,C)} &= t_{12}t_{23}(B,A,C) \\
&= t_{12}(B,C,A) \\
&= (C,B,A) \quad \checkmark \\
\\
\underline{t_{23}t_{12}t_{23}(A,B,C)} &= t_{23}t_{12}(A,C,B) \\
&= t_{23}(C,A,B) \\
&= (C,B,A) \quad \checkmark
\end{aligned}
\qquad
\begin{aligned}
&\overset{\text{representation}}{\Longrightarrow} && \rho(t_{12}t_{23}t_{12}) &= \rho(t_{23}t_{12}t_{23}) \\
&\overset{\rho \text{ abelian}}{\Longrightarrow} && \rho(t_{12})^2\rho(t_{23}) &= \rho(t_{23})^2\rho(t_{12}) \\
&\overset{\text{1-dim case}}{\Longrightarrow} && \rho(t_{23}) &= \rho(t_{12})
\end{aligned}$$

Express this with the k^{tk} element instead of the 3rd:



$$t_{12}t_{2k}t_{12} = t_{2k}t_{12}t_{2k}$$

$$\Rightarrow \quad \underline{\rho(t_{12}) = \rho(t_{2k})} \quad \forall k$$

We can again rewrite this with $1 \to 2,\ \ 2 \to k,\ \ k \to l$:

$$\begin{aligned} t_{2k}t_{kl}t_{2k} &= t_{kl}t_{2k}t_{kl} \\ \Rightarrow \rho(t_{2k}) &= \rho(t_{kl}) \quad \forall\ k,l \\ \Rightarrow \rho(t_{kl}) &= \rho(t_{12}) \quad \forall\ k,l \end{aligned}$$

More elegantly (in cycle notation):

$$\begin{aligned} (ij)(jk)(ij) &= (ij)(ikj) = (ik) \\ (jk)(ij)(jk) &= (jk)(jki) = (ki) \end{aligned}$$

Since 1-dimensional representations are abelian:

$$\Rightarrow \rho(t_{ij}t_{jk}t_{ij}) = \underbrace{\rho(t_{ij})^2}_{=1}\rho(t_{jk}) = \rho(t_{ik}) = \rho(t_{ij}) = \rho(t_{jk}t_{ij}t_{jk})$$

$$\text{and thus} \quad \underline{\rho(t_{kl})} = \rho(t_{kj}) = \underline{\rho(t_{ij})} \quad \forall i,j,k,l \in \{1,..,n\}$$

All transpositions must be mapped to the same number and since the transpositions generate S_n, the representation is completely determined by:

$$\text{either} \quad \rho(t_{kl}) \overset{(*)}{=} 1 \quad \forall\ k,l \quad \textbf{or} \quad \rho(t_{kl}) \overset{(**)}{=} -1 \quad \forall\ k,l$$

$(*)$: trivial representation

$(**)$: signum representation

<u>Trivial representation:</u> $\quad \rho(\pi) = \rho(\prod_{j=1}^{n} t_{k_j l_j}) = \prod_{j=1}^{n} \underbrace{\rho(t_{k_j l_j})}_{=1} = 1 \quad \forall \pi \in S_n$

<u>Signum representation:</u> $\quad \rho(\pi) = \rho(\prod_{j=1}^{n} t_{k_j l_j}) = \prod_{j=1}^{n} \underbrace{\rho(t_{k_j l_j})}_{=-1} = (-1)^n \equiv \mathrm{sgn}(\pi) \quad \forall \pi \in S_n$

Exercise 2

a) $P^2 = T^2 = \mathbb{1}$

V_2	$\mathbb{1}$	P	T	PT
$\mathbb{1}$	$\mathbb{1}$	P	T	PT
P	P	$\mathbb{1}$	PT	T
T	T	PT	$\mathbb{1}$	P
PT	PT	T	P	$\mathbb{1}$

b) $[g] = \{hgh^{-1} | h \in V_2\} \underset{\substack{\uparrow \\ h^{-1}=h \\ \text{abelian}}}{=} \{\overbrace{h^2}^{id} g | h \in V_2\} = \{g\}$

$\Rightarrow$ 4 conjungacy classes: $[\mathbb{1}]=\{\mathbb{1}\}$, $[P]=\{P\}$, $[T]=\{T\}$, $[PT]=\{PT\}$

c) H invariant subgroup: $ghg^{-1} \in H \quad \forall h \in H, \quad g \in G = V_2$

$\Rightarrow$ since V_2 abelian $\quad ghg^{-1} = h \in H \quad \forall g$

$\Rightarrow$ <u>every subgroup is invariant</u>

Subgroups: $\{\mathbb{1}\}$, $\underbrace{\{\mathbb{1}, P\}}_{P^2=\mathbb{1}}$, $\underbrace{\{\mathbb{1}, T\}}_{T^2=\mathbb{1}}$, $\underbrace{\{\mathbb{1}, PT\}}_{(PT)^2=\mathbb{1}}$

d) Regular representation:

$$\Gamma^{\text{reg}}_{ij}(g) = \delta(g_i^{-1} g g_j) \underset{\substack{\uparrow \\ g_i^{-1}=g_i \\ \text{abelian}}}{=} \delta(g_i^{-1} g_j g) \qquad \text{with} \quad \delta(g) = \begin{cases} 1, & g = \mathbb{1} \\ 0, & \text{else} \end{cases}$$

Les us arrange the group elements in a list:

$$V_2 = \{I,\ P,\ T,\ PT\} = \{g_0,\ g_1,\ g_2,\ g_3\}$$

$$\longrightarrow \qquad \Gamma^{\text{reg}}_{ij}(\mathbb{1}) = \underbrace{\delta(g_i g_j)}_{\substack{\neq 0 \text{ for } g_i=g_j \\ \Rightarrow i=j}} \overset{i=j}{\overset{\downarrow}{=}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \qquad \Gamma^{\text{reg}}_{ij}(P) = \underbrace{\delta(g_i g_j)P}_{\overset{!}{\underset{(*)}{=}} P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{array}{c} (*) \\ \mathbb{1}, P \;\rightarrow\; 0/1 \\ P, \mathbb{1} \;\rightarrow\; 1/0 \\ PT, T \;\rightarrow\; 3/2 \\ T, PT \;\rightarrow\; 2/3 \end{array}$$

$$\longrightarrow \qquad \Gamma^{\text{reg}}_{ij}(T) = \underbrace{\delta(g_i g_j)T}_{\overset{!}{\underset{(**)}{=}} T} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} (**) \\ \mathbb{1}, T \;\rightarrow\; 0/2 \\ T, \mathbb{1} \;\rightarrow\; 2/0 \\ PT, P \;\rightarrow\; 3/1 \\ P, PT \;\rightarrow\; 1/3 \end{array}$$

$$\longrightarrow \qquad \Gamma^{\text{reg}}_{ij}(PT) = \underbrace{\delta(g_i g_j)PT}_{\overset{!}{\underset{(***)}{=}} PT} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} (***) \\ \mathbb{1}, PT \;\rightarrow\; 0/3 \\ PT, \mathbb{1} \;\rightarrow\; 3/0 \\ P, T \;\rightarrow\; 1/2 \\ T, P \;\rightarrow\; 2/1 \end{array}$$

Exercise 3

a)

$$\left.\begin{array}{lll} P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} & \Rightarrow & P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} & \Rightarrow & T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \Rightarrow \; PT = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & & \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right\} \rho$$

This representation is reducible!

b) Character table of the representation of a):

V_2	$\mathbb{1}$	P	T	PT
ρ	2	0	0	-2

Character table of the irreducible representations of V_2:

1. # of irreducible reps. = # of cunjugation classes $\equiv n = 4$

2. dimensions of irreps. ρ_i: $|V_2| = \sum_{k=1}^{n} \mathrm{d}_k^2 = 4 \quad \Rightarrow \mathrm{d}_k = 1 \quad \forall k$

   $\Rightarrow$ 4 irreps. ρ_i of dimension 1.

3. neutral element: $\chi^{(k)}(\mathbb{1}) = \mathrm{d}_k = 1 \quad \forall k$

4. trivial representation: $\chi^{(1)}(g) = 1 \quad \forall g \in V_2$ ($\rho_1 \equiv$tivial rep.)

5. orthogonality of rows: $\sum_{k=1}^{n} g_k \chi^{\mu}(h_k)\chi^{o}(h_k) = g\delta_{\mu o}$

6. orthogonality of columns: $\sum_{\mu=1}^{n} \chi^{\mu}(h_k)\chi^{\mu}(h_{k'}) = \frac{g}{g_k}\delta_{kk'}$

$\Big\}\ h_k, h_k' \in V_2$

$$\Longrightarrow \begin{array}{l} g_k = \#\text{ of elements in conjugation class } [h_k] \\ g = \#\text{ of Group elements} \end{array} .$$

The character table looks as follows:

V_2	${}_1 1$	${}_1 P$	${}_1 T$	${}_1 PT$
ρ_1	1	1	1	1
ρ_2	1	x	y	z
ρ_3	1	x'	y'	z'
ρ_4	1	x''	y''	z''

$\rightarrow$ find the unknown elements!

<u>case 1)</u> rows 1 and 2 $\Rightarrow \quad \mu = 1,\ o = 2$

$$k = 1^4 \chi^{(1)}\overline{\chi^{(2)}} = 0 = 1 + x + y + z \quad (*)$$

<u>case 2)</u> compare row 2 with itself: $\quad (\mu = u = 2)$

$$\sum_{k=1}^{4} \chi^{(2)}\overline{\chi^{(2)}} = 1 + x^2 + y^2 + z^2 = 4\delta_{22} = 4$$

$$\Longrightarrow \quad x = \pm 1, \quad y = \pm 1, \quad z = \pm 1$$

$(*)$ allows the following pairs:

$$\underbrace{x = 1, \quad y = -1, \quad z = 1}_{\{x,y,z\}} \quad | \quad \underbrace{x = -1, \quad y = 1, \quad z = -1}_{\{x',y',z'\}} \quad | \quad \underbrace{x = -1, \quad y = -1, \quad z = 1}_{\{x'',y'',z''\}}$$

$\Rightarrow$ We have found all the possibilities and don't have to look further $\Rightarrow$ <u>character table</u>::

V_2	${}_1\mathbb{1}$	${}_1 P$	${}_1 T$	${}_1 PT$
ρ_1	1	1	1	1
ρ_2	1	1	-1	-1
ρ_3	1	-1	1	-1
ρ_4	1	-1	-1	1

c)

$$M \in V_2, \quad A \in O(1,1), \quad \Leftrightarrow \quad A^T g A = g, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P$$

$$\rightarrow \quad MA \in O(1,1)?$$

$$\rightarrow \quad (MA)^T g M A = A^T M^T g M A \underset{\substack{\uparrow \\ g=P \\ V_2 \text{ abelian}}}{=} A^T M^T M g A \underset{\substack{\uparrow \\ M^T=M \\ M^2=\mathbb{1}}}{=} A^T g A = \underline{g}$$

$\square$

d) $A^T g A = g$

$$\left.\begin{aligned} \Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right\} \Rightarrow \begin{cases} a^2 - c^2 = 1 & (1) \\ b^2 - d^2 = -1 & (2) \\ ab = cd & (3) \end{cases}$$

$\rightarrow$ 3 equations for 4 variables

$\Rightarrow$ If we're lucky we can construct a one-dimensional solution space

$\rightarrow$ Set $z := a$, $\boxed{\text{z >1}}$

$$\underset{(1)}{\Rightarrow} \quad z^2 - c^2 = 1 \quad \Rightarrow \boxed{c = \pm\sqrt{z^2 - 1}}$$

$$\underset{(3)}{\Rightarrow} \quad b = \frac{cd}{z} \quad \underset{(2)}{\Rightarrow} \quad \frac{c^2 d^2}{z^2} - d^2 = d^2 \left(\frac{c^2}{z^2} - 1 \right) = -1$$

$$\Rightarrow \quad d^2 = \frac{-1}{\frac{c^2}{z^2} - 1} = \frac{-1}{\frac{\cancel{z^2}-1}{z^2} - \frac{\cancel{z^2}}{z^2}} = z^2$$

$$\Rightarrow \quad \boxed{d = \pm z}$$

$$\Rightarrow \quad \boxed{b = \frac{cd}{z} = \frac{\pm\sqrt{z^2 - 1}(\pm z)}{z} = \pm\sqrt{z^2 - 1}}$$

with $\boxed{sign(b) = sign(c) \cdot sign(d)}$ (*)

$$\Rightarrow \quad A = \begin{pmatrix} z & \pm\sqrt{z^2 - 1} \\ \pm\sqrt{z^2 - 1} & \pm z \end{pmatrix}$$

e) $\tilde{A} = \begin{pmatrix} z & \pm\sqrt{z^2-1} \\ \pm\sqrt{z^2-1} & \pm z \end{pmatrix}$

Different possibilities for *sign*'s:

$$(sign(b) = sign(c) \cdot sign(d), \quad \text{from } (*))$$

$$A_0 = \begin{pmatrix} z & \sqrt{1-z^2} \\ \sqrt{1-z^2} & z \end{pmatrix}, \qquad A2 = \begin{pmatrix} z & -\sqrt{1-z^2} \\ \sqrt{1-z^2} & -z \end{pmatrix}$$

$$A_1 = \begin{pmatrix} z & -\sqrt{1-z^2} \\ -\sqrt{1-z^2} & z \end{pmatrix}, \qquad A3 = \begin{pmatrix} z & \sqrt{1-z^2} \\ -\sqrt{1-z^2} & -z \end{pmatrix}$$

Now let V_2 act upon $\tilde{A}$:

$$\underline{\mathbb{1}\tilde{A}} = \tilde{A} \equiv \underline{A_0}$$

$$\underline{P\tilde{A}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{A} = \begin{pmatrix} z & \sqrt{1-z^2} \\ -\sqrt{1-z^2} & -z \end{pmatrix} \equiv \underline{A_3}$$

$$\underline{T\tilde{A}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{A} = \begin{pmatrix} -z & -\sqrt{1-z^2} \\ \sqrt{1-z^2} & z \end{pmatrix} \underset{z\mapsto -z}{\hat{=}} \begin{pmatrix} z & -\sqrt{1-z^2} \\ \sqrt{1-z^2} & -z \end{pmatrix} \equiv \underline{A_2}$$

$$\underline{PT\tilde{A}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{A} = \begin{pmatrix} -z & -\sqrt{1-z^2} \\ -\sqrt{1-z^2} & -z \end{pmatrix} \underset{z\mapsto -z}{\hat{=}} \begin{pmatrix} z & -\sqrt{1-z^2} \\ -\sqrt{1-z^2} & z \end{pmatrix} \equiv \underline{A_1}$$

$$\Rightarrow \underline{\left\{ M\tilde{A} \quad | M \in V_2 \right\} \text{ spans } O(1,1)}$$

f)

$$\begin{aligned} z &= \cosh\lambda \\ z^2 - 1 &= \sinh^2\lambda \end{aligned} \quad \Rightarrow \boxed{\tilde{A} = \begin{pmatrix} \cosh\lambda & |\sinh\lambda| \\ |\sinh\lambda| & \cosh\lambda \end{pmatrix}} \quad \lambda \in \mathbb{R}$$