Exercise 1. ***Topological charge***

Consider a $SU(N)$ gauge theory with covariant derivative defined as

$$D_\mu = \partial_\mu + igA_\mu, \quad A_\mu = A^a_\mu T^a, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\,, \tag{1}$$

where the T^a are the generators of $SU(N)$, which satisfy $[T^a, T^b] = if^{abc}T^c$.

1. Show that $F^a_{\mu\nu}\tilde{F}^{\mu\nu,a} = \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$ can be written as a total derivative, namely

$$F^a_{\mu\nu}\tilde{F}^{\mu\nu,a} = \partial_\mu K^\mu\,, \qquad K^\mu = \epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} + \frac{g}{3}f_{abc}A^a_\nu A^b_\rho A^c_\sigma\right)\,. \tag{2}$$

2. The topological charge is defined as

$$n = \frac{g^2}{32\pi^2}\int \mathrm{d}^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu,a} = \frac{g^2}{32\pi^2}\int \mathrm{d}^3x\, K_0\Big|_{t=-\infty}^{t=+\infty}\,. \tag{3}$$

   Consider an adiabatic transformation of the type

$$\begin{aligned} A_\mu(t=-\infty) &= 0 \\ A_\mu(t=+\infty) &= \frac{i}{g}\left(\partial_\mu\Lambda\right)\Lambda^{-1}\,, \qquad \text{with} \qquad \Lambda = \frac{\mathbf{x}^2-d^2}{\mathbf{x}^2+d^2}\mathbf{1} + i\frac{2d}{\mathbf{x}^2+d^2}x_k\tau_k\,, \end{aligned} \tag{4}$$

   where τ_i are the generators of a $SU(2)$ subgroup, $\tau_i\tau_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\tau_k$.
   Check that $F_{\mu\nu} = 0$ and, hence, that the topological charge becomes

$$n = i\frac{g^3}{24\pi^2}\int \mathrm{d}^3x\,\epsilon^{ijk}\,\mathrm{Tr}\,\mathbf{A}_i\mathbf{A}_j\mathbf{A}_k\,. \tag{5}$$

3. Show that $A_0(t=\infty) = 0$ and

$$A_i(t=+\infty) = \frac{-2\,d}{g(\mathbf{x}^2+d^2)^2}\left[(\mathbf{x}^2-d^2)\tau_i - 2(x_j\tau_j)x_i + 2d\,\epsilon_{ijk}x_j\tau_k\right]\,. \tag{6}$$

4. Compute the topological charge n of this adiabatic transformation.
   (Hint: recall the contraction identities $\epsilon^{ijk}\epsilon^{ijm} = 2\delta^{km}$ and $\epsilon^{ijk}\epsilon^{iml} = \delta^{jm}\delta^{kl} - \delta^{jl}\delta^{km}$).