**Exercise 1:** Space-time rotations

a) Show that the euclidean norm of a 3-vector is preserved under a rotation of the reference frame around the z-axis.

b) A Lorentz transformation $\Lambda$ is a space-time coordinates transformation and it is implemented by 4x4 matrices that preserve the norm $g = \text{diag}(1, -1, -1, -1)$, and not the euclidean norm:

$$O(3, 1) = \{ \Lambda \in \text{Mat}(4, \mathbb{R}) | \Lambda^T g \Lambda = g \}$$ (1)

Show that it is the case, given the Lorentz boost $\Lambda$ in the z-direction defined as:

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta \gamma & 0 & 0 & \gamma \end{pmatrix}$$ (2)

**Exercise 2:** Lorentz invariant measure

Show that

$$\frac{d^3 \vec{p}}{2p_0}$$

is Lorentz invariant

Hint: implement Einstein’s dispersion relation in a $\delta$ to fix the relation between time and spatial components of a 4-vector

**Exercise 3:** Circular accelerators

In a circular accelerator two massless particles are colliding head to head with momenta

$$p_1^\mu = E_1(1, 0, 0, 1)$$

$$p_2^\mu = E_2(1, 0, 0, -1)$$ (3)

in the laboratory frame.

a) After drawing a schematic representation of the process, write down the Lorentz transformation that connects the laboratory frame to the center of mass frame.
b) A particle of mass \( m = 10 \text{ GeV} \) is produced in a head to head collision between two massless particles of energy \( E = 5.01 \text{ GeV} \) each. Determine the boost of the particle with respect to the laboratory frame.

c) Use the result from point (a) to determine the boost with respect to the laboratory frame if the two colliding particles have energies \( E_1 = 3.1 \text{ GeV} \) and \( E_2 = 9.0 \text{ GeV} \), respectively.

**Exercise 4:** Lorentz invariants at colliders

The transverse momentum \( \vec{p}_T \) is defined as the sum of the spatial components of the 4-momentum which are perpendicular to the beam axis, while the rapidity \( y \) is defined as:

\[
y = \frac{1}{2} \log \left( \frac{E + p_z}{E - p_z} \right)
\] (4)

Where \( E \) and \( p_z \) are the zero-th and third components of the 4-momentum, respectively.

a) Show that, given a generic 4-momentum

\[
p^\mu = (E, |\vec{p}| \cos \varphi \sin \theta, |\vec{p}| \sin \varphi \sin \theta, |\vec{p}| \cos \theta)
\] (5)

where \( \theta \) is the angle between \( \vec{p} \) and the z-axis and \( \varphi \) is the angle between \( \vec{p}_T \) and the x-axis, \( \vec{p}_T \) and \( \Delta y = y_1 - y_2 \) are Lorentz invariant quantities given a boost along the z-axis.

b) Show that for a particle moving in the z direction

\[
\cosh y = \gamma \quad \text{and} \quad \sinh y = \gamma \beta
\]

**Exercise 5:** Madelstam variables

Given a generic 4-body process

\[
p_1 + p_2 = p_3 + p_4
\]

With massless particles (i.e. \( p_i^2 = 0 \) for each \( i \)), show that it can be described by only two Lorentz invariant quantities.