Exercise 1. ***Dimensional regularisation***

Prove the following identities:

$$\begin{aligned}\int \frac{\mathrm{d}^d p_E}{(2\pi)^d}\frac{1}{(p_E^2+\Delta)^n} &= \frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\,\Delta^{\frac{d}{2}-n}\,,\\ \int \frac{\mathrm{d}^d p_E}{(2\pi)^d}\frac{p_E^2}{(p_E^2+\Delta)^n} &= \frac{1}{(4\pi)^{d/2}}\,\frac{d}{2}\,\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\,\Delta^{1+\frac{d}{2}-n}\,.\end{aligned} \tag{1}$$

Solution. Let us consider the integral

$$I_\alpha = \int \frac{\mathrm{d}^d p_E}{(2\pi)^d}\frac{(p_E^2)^\alpha}{(p_E^2+\Delta)^n}. \tag{S.1}$$

We observe that the integrand depends on p_E^2 only. Therefore, we can introduce d-dimensional spherical coordinates and perform the angular integration,

$$\int \mathrm{d}^d p_E f(p_E^2) = \frac{1}{2}\int \mathrm{d}\,p_E^2 (p_E^2)^{d/2-1}\mathrm{d}^{d-1}\Omega_d f(p_E^2) = \frac{1}{2}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty \mathrm{d}\,p_E^2 (p_E^2)^{d/2-1} f(p_E^2)\,. \tag{S.2}$$

By applying eq. (S.2) to the definition of I_α, we obtain the one-dimensional integral

$$I_\alpha = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty \mathrm{d}p_E^2 \frac{(p_E^2)^{\alpha+d/2-1}}{(p_E^2+\Delta)^n}, \tag{S.3}$$

which we can transform to the integral representation of the Beta function through the change of variables

$$x = \frac{p_E^2}{p_E^2+\Delta}\,, \tag{S.4}$$

which maps the integration bounds to $0 \le x \le 1$ and implies

$$p_E^2 = \Delta\frac{x}{1-x}, \quad p_E^2+\Delta = \Delta\frac{1}{1-x}, \quad \mathrm{d}p_E^2 = \Delta\frac{1}{(1-x)^2}\mathrm{d}x\,. \tag{S.5}$$

In this way, we arrive at

$$I_\alpha = \frac{\Delta^{\alpha+d/2-n}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^1 \mathrm{d}x \left(\frac{x}{1-x}\right)^{\alpha+d/2-1}(1-x)^{n-2} \tag{S.6}$$

$$= \frac{\Delta^{\alpha+d/2-n}}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\, x^{\alpha+d/2-1}(1-x)^{n-\alpha-d/2-1} \tag{S.7}$$

$$= \frac{\Delta^{\alpha+d/2-n}}{(4\pi)^{d/2}}\frac{\Gamma(\alpha+d/2)\Gamma(n-\alpha-d/2)}{\Gamma(n)\Gamma(d/2)}. \tag{S.8}$$

By specialising the previous result to $\alpha = 0, 1$, we obtain

$$\begin{aligned} I_0 &= \int \frac{\mathrm{d}^d p_E}{(2\pi)^d}\frac{1}{(p_E^2+\Delta)^n} = \frac{\Delta^{d/2-n}}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\,,\\ I_1 &= \int \frac{\mathrm{d}^d p_E}{(2\pi)^d}\frac{p_E^2}{(p_E^2+\Delta)^n} = \frac{d}{2}\,\frac{\Delta^{1+d/2-n}}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2-1)}{\Gamma(n)}\,,\end{aligned}$$

where, in the last equality, we have used $\Gamma(z+1) = z\Gamma(z)$.

Exercise 2. ***Feynman parametrisation***

Prove by induction the formula for the Feynman parametrisation of n propagators:

$$\frac{1}{D_1^{a_1}\dots D_n^{a_n}} = \frac{\Gamma(a_1+a_2+\dots+a_n)}{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_n)} \times \int_0^1 \mathrm{d}x_1 \dots \int_0^1 \mathrm{d}x_n \frac{\delta(1-x_1-x_2-\dots-x_n)\; x_1^{a_1-1}x_2^{a_2-1}\dots x_n^{a_n-1}}{[x_1D_1+\dots+x_nD_n]^{a_1+\dots+a_n}} \,. \tag{2}$$

Solution. In order to prove eq. (2) by induction, we first need to verify the case $n=2$,

$$\frac{1}{D_1^{a_1}D_2^{a_2}} = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \frac{\delta(1-x_1-x_2)\; x_1^{a_1-1}x_2^{a_2-1}}{[x_1D_1+x_2D_2]^{a_1+a_2}} \,, \tag{S.9}$$

and subsequently show that, if eq. (2) holds for arbitrary n, then it also holds for $n+1$.

- $n=2$: The proof of eq. (S.9) is obtained by rewriting the integral on the r.h.s. in terms of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$. In fact,

$$\begin{aligned}
&\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 \mathrm{d}x_1\mathrm{d}x_2 \frac{x_1^{a_1-1}x_2^{a_2-1}}{(x_1D_1+x_2D_2)^{a_1+a_2}} \delta\left(1-x_1-x_2\right) \\
&\quad = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 \mathrm{d}x_1 \frac{x_1^{a_1-1}(1-x_1)^{a_2-1}}{D_2^{a_1+a_2}\left(1-\frac{D_2-D_1}{D_2}x_1\right)^{a_1+a_2}} \\
&\quad = D_2^{-a_1-a_2}\; {}_2F_1\left(a_1+a_2, a_1, a_1+a_2; \frac{D_2-D_1}{D_2}\right) \\
&\quad = D_2^{-a_1-a_2}\left(\frac{D_1}{D_2}\right)^{-a_1} = \frac{1}{D_1^{a_1}D_2^{a_2}} \,.
\end{aligned} \tag{S.10}$$

  where, in the third equality, we have made use of

$${}_2F_1\left(a_1+a_2, a_1, a_1+a_2; z\right) = (1-z)^{-a_1} \,, \tag{S.11}$$

  with $z=(D_2-D_1)/D_2$. This identity can be verified by comparing the defining series of the hypergeometric function,

$$\begin{aligned}
{}_2F_1\left(a_1+a_2, a_1, a_1+a_2; z\right) &= \sum_{n=0}^{\infty} \frac{(a_1+a_2)_n(a_1)_n}{(a_1+a_2)_n}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)}{\Gamma(a_1)}\frac{z^n}{n!} \,,
\end{aligned} \tag{S.12}$$

with the Taylor expansion of $(1-z)^{-a_1}$ around $z=0$,

$$\begin{aligned}(1-z)^{-a_1} &= \sum_{n=0}^{\infty} \frac{d^n}{dz^n}(1-z)^{-a_1}\Big|_{z=0} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} (-a_1)(-a_1-1)\cdots(-a_1-n+1)(-1)^n (1-z)^{-a_1-n}\Big|_{z=0} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} (a_1)(a_1+1)\cdots(a_1+n-1)\Big|_{z=0} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)}{\Gamma(a_1)} \frac{z^n}{n!}\,. \end{aligned} \tag{S.13}$$

- $n+1$: We can now turn to the induction step. By assuming the Feynman parametrisation to hold for the case of n denominators, we write

$$\frac{1}{\left(\prod_{i=1}^n D_i^{a_i}\right) D_{n+1}^{a_{n+1}}} = \frac{1}{D_{n+1}^{a_{n+1}}} \frac{\Gamma(\sum_{i=1}^n a_i)}{\prod_{i=1}^n \Gamma(a_i)} \int_0^1 \left(\prod_{i=1}^n \mathrm{d}x_i\right) \frac{\delta\left(1-\sum_{i=1}^n x_i\right) \prod_{i=1}^n x_i^{a_i-1}}{(x_1 D_1 + x_2 D_2 + \cdots x_n D_n)^{\sum_{i=1}^n a_i}}\,. \tag{S.14}$$

Next, we introduce two additional parameters, u and v, and use eq. (S.9) to combine all the denominators

$$\begin{aligned}\frac{1}{\left(\prod D_i^{a_i}\right) D_{n+1}^{a_{n+1}}} &= \frac{\Gamma(\sum_{i=1}^n \sum a_i + a_{n+1})\Gamma(\sum_{i=1}^n a_i)}{\Gamma(a_{n+1})\Gamma(\sum_{i=1}^n a_i)\prod_{i=1}^n \Gamma(a_i)} \int_0^1 \mathrm{d}u\mathrm{d}v\delta(1-u-v) u^{a_{n+1}-1} v^{\sum_{i=1}^n a_i - 1} \\ \times \int_0^1 \left(\prod_{i=1}^n \mathrm{d}x_i\right) \delta\left(1-\sum_{i=1}^n x_i\right) \prod_{i=1}^n x_i^{a_i-1} & \frac{1}{(uD_{n+1} + v(x_1 D_1 + \cdots x_n D_n))^{\sum_{i=1}^n a_i + a_{n+1}}}\,. \end{aligned} \tag{S.15}$$

We can now bring the denominator back to the standard form by rescaling the Feynman parameters x_i as

$$y_i = v x_i\,, \qquad \prod_{i=1}^n dx_i = v^{-n} \prod_{i=1}^n dy_i\,, \tag{S.16}$$

in such a way to obtain

$$\begin{aligned}\frac{1}{\left(\prod_{i=1}^n D_i^{a_i}\right) D_{n+1}^{a_{n+1}}} &= \frac{\Gamma(\sum_{i=1}^{n+1} \sum a_i)}{\prod_{i=1}^{n+1} \Gamma(a_i)} \int_0^1 \mathrm{d}u\mathrm{d}v\, \delta(1-u-v) u^{a_{n+1}-1} v^{\sum_{i=1}^n a_i - 1 - n - \sum_{i=1}^n (a_i-1)} \\ &\quad \times \int_0^v \left(\prod_{i=1}^n \mathrm{d}y_i\right) \delta\left(1 - \frac{1}{v}\sum_{i=1}^n y_i\right) \frac{\prod_{i=1}^n y_i^{a_i-1}}{(uD_{n+1} + y_1 D_1 + \cdots y_n D_n))^{\sum_{i=1}^{n+1} a_i}} \end{aligned} \tag{S.17}$$

$$\begin{aligned}&= \frac{\Gamma(\sum_{i=1}^{n+1} \sum a_i)}{\prod_{i=1}^{n+1} \Gamma(a_i)} \int_0^1 \mathrm{d}u\mathrm{d}v\, \delta(1-u-v) u^{a_{n+1}-1} \\ &\quad \times \int_0^v \left(\prod_{i=1}^n \mathrm{d}y_i\right) \delta\left(v - \sum_{i=1}^n y_i\right) \frac{\prod_{i=1}^n y_i^{a_i-1}}{(uD_{n+1} + y_1 D_1 + \cdots y_n D_n)^{\sum_{i=1}^{n+1} a_i}}\,, \end{aligned} \tag{S.18}$$

where, in the second equality, we have used $\delta\left(1-\frac{1}{v}\sum_{i=1}^{n} y_i\right) = v\;\delta\left(v-\sum_{i=1}^{n} y_i\right)$ to get rid of all v factors. This δ-function can be used to evaluate the integral over v. In this way, by relabelling $u = y_{n+1}$, we arrive at

$$\frac{1}{\left(\prod_{i=1}^{n} D_i^{a_i}\right) D_{n+1}^{a_{n+1}}} = \frac{\Gamma(\sum_{i=1}^{n+1}\sum a_i)}{\prod_{i=1}^{n+1}\Gamma(a_i)} \int_0^1 \left(\prod_{i=1}^{n+1} \mathrm{d}y_i\right) \frac{\delta(1-\sum_{i=1}^{n+1} y_i)\prod_{i=1}^{n+1} y_i^{a_i-1}}{(y_1 D_1 + \cdots y_{n+1}D_{n+1})^{\sum_{i=1}^{n+1} a_i}}\,, \tag{S.19}$$

which exactly corresponds to the Feynman parametrization formula of eq. (2) for $n+1$ denominators. Notice that, after integrating over v, we have extended the integral of each y_i over the interval $0 \le y_i \le 1$, since, due the remaining of the δ-function, all values $y_i > 1 - y_{n+1}$ do not contribute to the integral.

Exercise 3. *Generalisation of γ_5 in d dimensions*

In four dimensions, γ_5 can be defined by its two properties:

$$\begin{aligned} &(a) \qquad \mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) = \epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,\mathbf{1}\,, \\ &(b) \qquad \{\gamma_\mu, \gamma_5\} = 0\,. \end{aligned} \tag{3}$$

Show that those two properties cannot be maintained simultaneously in d dimensions. Assume Tr(. . .) *to be a meromorphic function of the dimension d, then show that (b) leads to*

$$\begin{aligned} \mathrm{Tr}\left(\gamma_5\right) &= 0\,, \\ \mathrm{Tr}\left(\gamma_5\gamma_\mu\gamma_\nu\right) &= 0\,, \\ \mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) &= 0\,. \end{aligned} \tag{4}$$

Solution. The extension of γ-matrices to d dimensions affects contraction identities, such as

$$\begin{aligned} &(c) \qquad \gamma^\alpha\gamma_\alpha = \frac{1}{2}\{\gamma^\alpha, \gamma_\alpha\} = g^\alpha_\alpha = d\,, \\ &(d) \qquad \gamma^\alpha\gamma_\mu\gamma_\alpha = \gamma^\alpha\left(-\gamma_\alpha\gamma_\mu + 2g_{\alpha\mu}\right) = -(d-2)\gamma_\mu\,, \\ &(e) \qquad \gamma^\alpha\gamma_\mu\gamma_\nu\gamma_\alpha = \gamma^\alpha\gamma_\mu\left(-\gamma_\alpha\gamma_\nu + 2g_{\nu\alpha}\right) = (d-2)\gamma_\mu\gamma_\nu + 2\gamma_\nu\gamma_\mu \\ &\qquad\qquad = (d-2)\gamma_\mu\gamma_\nu - 2\gamma_\mu\gamma_\nu + 4g_{\mu\nu} = (d-4)\gamma_\mu\gamma_\nu + 4g_{\mu\nu}\,. \end{aligned} \tag{S.20}$$

These identities, together with (b), can be used in oder to evaluate traces involving γ_5. Let's start with γ_5:

$$d\;\mathrm{Tr}\left(\gamma_5\right) = \mathrm{Tr}\left(\gamma^5\gamma_\alpha\gamma^\alpha\right) = -\,\mathrm{Tr}\left(\gamma^\alpha\gamma^5\gamma_\alpha\right) = -d\;\mathrm{Tr}\left(\gamma^5\right)\,, \tag{S.21}$$

where we have used, in order, the identity (c), the anticommutation rule (b), the cyclicity of the trace and finally (c) again. We see that for all $d \neq 0$, we have $\mathrm{Tr}(\gamma_5) = 0$, therefore we have to choose

$$\mathrm{Tr}\left(\gamma^5\right) = 0\,, \tag{S.22}$$

if we want to keep the trace meromorphic. The other two identities are derived in a similar fashion:

- In the case of two γ-matrices, we have

$$\begin{aligned} d\ \mathrm{Tr}\left(\gamma_5\gamma_\mu\gamma_\nu\right) &= \mathrm{Tr}\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma^\alpha\right) = -\,\mathrm{Tr}\left(\gamma_5\gamma^\alpha\gamma_\mu\gamma_\nu\gamma_\alpha\right) \\ &= -\,(d-4)\,\mathrm{Tr}\left(\gamma_5\gamma_\mu\gamma_\nu\right) - 4g^{\mu\nu}\,\mathrm{Tr}\left(\gamma_5\right) = -\,(d-4)\,\mathrm{Tr}\left(\gamma_5\gamma_\mu\gamma_\nu\right)\,, \end{aligned} \tag{S.23}$$

where we have used eq. (S.22) in the last equality. In this way, we obtain

$$(d-2)\,\mathrm{Tr}\left(\gamma_5\gamma_\mu\gamma_\nu\right) = 0\,, \tag{S.24}$$

which, for arbitrary d, requires

$$\mathrm{Tr}\left(\gamma_5\gamma_\mu\gamma_\nu\right) = 0\,. \tag{S.25}$$

- Finally, in the case of four γ-matrices, we have

$$\begin{aligned} (d-4)\,\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) &= \mathrm{Tr}\left(\gamma_\alpha\gamma_\mu\gamma_\nu\gamma^\alpha\gamma_\rho\gamma_\sigma\gamma_5\right) - 4g^{\mu\nu}\,\mathrm{Tr}\left(\gamma_\rho\gamma_\sigma\gamma_5\right) \\ &= -\,\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma^\alpha\gamma_\rho\gamma_\sigma\gamma_\alpha\gamma_5\right) \\ &= -(d-4)\,\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) - 4g^{\rho\sigma}\,\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_5\right) \\ &= -(d-4)\,\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right)\,, \end{aligned} \tag{S.26}$$

where, in the second and last equality, we have used eq. (S.25). Hence, we have obtained

$$(d-4)\,\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) = 0\,, \tag{S.27}$$

which is in contrast with (a).

Exercise 4. ***An explicit*** $1/\epsilon^2$ ***pole***

Calculate the diagram in Figure 1 in dimensional regularisation to see that it has a pole of second order at $d = 4$.

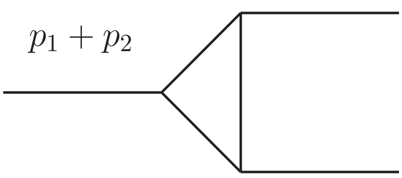

Figure 1: The one-loop vertex diagram in ϕ^3 theory.

Solution. We need to compute the one-loop diagram

$$I(Q^2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k)^2(k-p_2)^2(k+p_1)^2}\,, \tag{S.28}$$

where we have indicated its dependence on $Q^2 = (p_1+p_2)^2$. First, we introduce the Feynman parametrisation given in eq. (2) and we use the δ-function to integrate over the parameter x_1,

$$\begin{aligned} I(Q^2) =& \frac{\Gamma(3)}{\Gamma(1)^3} \int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\delta(1-x_1-x_2-x_3)}{\left[x_1k^2 + x_2(k-p_2)^2 + x_3(k+p_1)^2\right]^3} \\ =& 2\int_0^1 \mathrm{d}x_2 \int_0^{1-x_2} \mathrm{d}x_3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2 - 2x_2 k\cdot p_2 + 2x_3 k\cdot p_1\right]^3}\,. \end{aligned} \tag{S.29}$$

Next, we cast the denominator in standard form through the change of variables

$$k'^{\mu} = k^{\mu} - x_2 p_2^{\mu} + x_3 p_1^{\mu}\,, \tag{S.30}$$

which leads to

$$I(Q^2) = 2\int_0^1 \mathrm{d}x_2 \int_0^{1-x_2} \mathrm{d}x_3 \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \frac{1}{\left[k'^2 - \Delta\right]^3}\,, \tag{S.31}$$

with $\Delta = -Q^2 x_2 x_3$. We can now perform a Wick rotation and transform the integration over the loop momentum into an Euclidean integral, which we can then evaluate through eq. (1),

$$\begin{aligned} I(Q^2) =& -2i\int_0^1 \mathrm{d}x_2 \int_0^{1-x_2} \mathrm{d}x_3 \int \frac{\mathrm{d}^d k'_E}{(2\pi)^d} \frac{1}{\left[k'^2_E + \Delta\right]^3} \\ =& \frac{-2i}{(4\pi)^{d/2}} \frac{\Gamma\left(3-\frac{d}{2}\right)}{\Gamma(3)} \int_0^1 \mathrm{d}x_2 \int_0^{1-x_2} \mathrm{d}x_3 \Delta^{\frac{d}{2}-3} \\ =& \frac{-i}{(4\pi)^{d/2}} \Gamma\left(3-\frac{d}{2}\right) (-Q^2)^{\frac{d}{2}-3} \int_0^1 \mathrm{d}x_2\, x_2^{\frac{d}{2}-3} \int_0^{1-x_2} \mathrm{d}x_3\, x_3^{\frac{d}{2}-3}\,. \end{aligned} \tag{S.32}$$

Note that the momentum integration led to a finite result in $d = 4$ so that, if present, divergencies must originate from the integral over the Feynman parameters. The latter returns

$$\begin{aligned} I(Q^2) =& \frac{-i}{(4\pi)^{d/2}} \Gamma\left(3-\frac{d}{2}\right) (-Q^2)^{\frac{d}{2}-3} \frac{2}{d-4} \int_0^1 \mathrm{d}x_2\, x_2^{d/2-3} (1-x_2)^{d/2-2} \\ =& \frac{-i}{(4\pi)^{d/2}} \Gamma\left(3-\frac{d}{2}\right) (-Q^2)^{\frac{d}{2}-3} \frac{2}{d-4} B\left(\frac{d}{2}-2, \frac{d}{2}-1\right)\,, \\ =& \frac{-i}{(4\pi)^{d/2}} (-Q^2)^{\frac{d}{2}-3} \frac{\Gamma^2\left(\frac{d-2}{2}\right)\Gamma\left(\frac{6-d}{2}\right)}{\Gamma(d-3)} \left(\frac{2}{d-4}\right)^2\,, \\ =& \frac{-i}{(4\pi)^{d/2}} (-Q^2)^{\frac{d}{2}-3} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \frac{1}{\epsilon^2}\,, \end{aligned} \tag{S.33}$$

where, in the last step, we have set $d = 4-2\epsilon$. This calculation shows that $I(Q^2)$ is quadratically divergent in $d = 4$ dimensions.