Global anomalies in the Standard Model(s) and Beyond

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Outline of talk

1. Motivation
2. Global anomalies and bordism (via the η-invariant)
3. Global anomalies in the Standard Models + BSM
4. Anomaly interplay in $U(2)$ gauge theories

Motivation

Standard Model (SM) successfully explains all data from collider experiments.

But...

- Dark matter?
- Dark energy?
- Neutrino oscillations?
- Matter-antimatter asymmetry?
- ...
- Flavour puzzle?
- Hierarchy problems?
- Physics beyond Planck scale?
- ...

... Need to go Beyond the Standard Model (BSM)

The SM is also not unique.

The SM gauge group G is ambiguous:

- Gauge boson interactions only determine Lie algebra of G to be $\mathfrak{g} = \mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$
- There are 4 groups with this Lie algebra that admit SM fermion representations:[1]

$$G = \frac{SU(3) \times SU(2) \times U(1)}{\Gamma_n}, \qquad \Gamma_n \cong \mathbf{1}, \mathbb{Z}_2, \mathbb{Z}_3, \text{ or } \mathbb{Z}_6, \qquad (1)$$

Γ_6 generated by $\omega = (e^{2\pi i/3}\mathbf{1}_3, -\mathbf{1}_2, e^{2\pi i/6})$;
Γ_3 by ω^2;
Γ_2 by ω^3.

[1] Assuming G is connected

$$G = \frac{SU(3) \times SU(2) \times U(1)}{\Gamma_n}, \quad \Gamma_n \cong \mathbf{1},\ \mathbb{Z}_2,\ \mathbb{Z}_3,\ \text{or } \mathbb{Z}_6 \qquad (2)$$

Could we tell the difference?

In theory – yes.[2]

1. Different periodicity of hypercharge θ angle
2. Different spectra of Wilson and 't Hooft line operators[3]
3. GUTs prefer the $\mathbb{Z}_6$ option

... with current experiments?
No

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[2]Tong, 1705.01853

[3]See Aharony, Seiberg, Tachikawa, 2013.

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... with current experiments?

No – unless LHC discovered new particles in representations that kill one of more of the options, e.g. $\phi \sim (\mathbf{1}, \mathbf{2})_{\frac{\text{even number}}{6}}$ or $\psi \sim (\mathbf{1}, \mathbf{1})_{\frac{\text{odd number}}{6}}$

$$G = \frac{SU(3)\times SU(2)\times U(1)}{\Gamma_n}, \quad \Gamma_n \cong \mathbf{1},\ \mathbb{Z}_2,\ \mathbb{Z}_3,\ \text{or } \mathbb{Z}_6, \tag{4}$$

Another possibility is that the four different SM gauge groups suffer from different anomalies.

- Perturbative anomalies automatically cancel for all four SMs
- ... but could be subtle global anomalies associated with topology of G. Perhaps not all four SMs are truly anomaly free?

Global anomalies in any of the 4 SMs?

Quick answer: No global anomalies in any of the SMs for the *specific* SM field content.[4] Reasoning: no global anomalies in 4d $SU(5)$ GUT

More refined answer: in *any* 4d G_{SM}/Γ_n gauge theory, there is at most[†] the Witten $SU(2)$ anomaly.[5] Cancelling this requires an even number of fermions with $j = 2r + 1/2$, $r \in \mathbb{Z}$.
Result holds if extend SM by arbitrary BSM matter fields.

Also considered popular extensions of the SM gauge group, and find no new global anomalies.

[†]No Witten anomaly in the Γ_2 or Γ_6 case, where $G_{\text{EW}} = U(2)$, due to an interplay between local and global anomalies.[6]

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[4] I. Garcia-Etxebarria and M. Montero, 2018, also D. Freed, 2007.
[5] JD, B. Gripaios, N. Lohitsiri, 1910.11277, also Z. Wan and J. Wang, 1910.14668.
[6] JD and N. Lohitsiri, 2001.07731.

Global anomalies, the η-invariant, and bordism

Ingredients for a chiral gauge theory

Let spacetime be a Euclidean 4-manifold Σ. We then need the following:

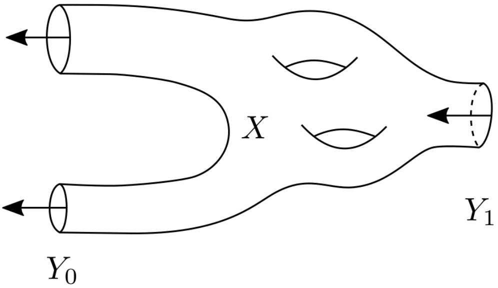
1. An orientation on Σ (SM breaks CP and thus breaks time-reversal)
2. A form of spin structure on Σ to define fermions,
3. A principal G-bundle over Σ to define gauge fields. Equivalently, a map $f : \Sigma \to BG$. 'B' means classifying space
4. A Dirac operator $i\not{D}$ which couples fermions to gauge fields

Assume theory defined on **all** 4-manifolds admitting these structures.

Bordism

Bordism is an equivalence between (smooth, compact, closed) mfds with these structures. Two d-mfds are bordant if exists a $d+1$-mfd X, with any 'structures' extended to X, such that

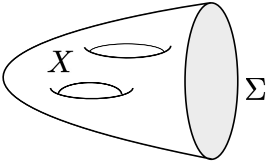
$$\partial X = Y_0 \sqcup (-Y_1), \tag{5}$$



Bordism partitions spin d-mfds with maps to BG into equivalence classes, which form an (abelian) group $\Omega_d^{\mathrm{Spin}}(BG)$ under disjoint union.

Bordism

E.g. the zero element in $\Omega_d^{\mathrm{Spin}}(BG)$ therefore contains all d-mfds which are boundaries of $d+1$-mfds, with spin structure & maps to BG extended.



We will need the concept of bordism shortly...

Fermionic partition functions

Anomalies can arise from the functional integration over fermions:

$$Z_\psi[A,\Sigma] \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{-\int_\Sigma d^4x\ \bar{\psi} i \not{D}\psi} = \det\, i\not{D},^{7} \tag{6}$$

- Non-anomalous: $Z_\psi[A,\Sigma]$ a $\mathbb{C}$-**function** on space of background data (e.g. on space of connections modulo gauge transformations).
- Anomalous: $Z_\psi[A,\Sigma]$ at best a **section** of a $\mathbb{C}$-bundle over the space of background data

[7] More generally, det $\to$ Pfaffian (if no conserved "chiral" charges)

Local anomaly:[8]

$Z_\psi[A] \neq Z_\psi[A^g]$ for $A \to A^g$ with $g \approx \mathbf{1}$. Seen by 1-loop triangle diagrams

Global anomaly:[9] any anomaly that is not local!

Example (Witten): 4d $SU(2)$ gauge theory with one fermion doublet, $Z_\psi[A] = -Z_\psi[A^U]$, for $U(x)$ in non-trivial class of $\pi_4(SU(2)) = \mathbb{Z}_2$

Global anomalies:

- Cannot be seen perturbatively (invisible in weak background fields)
- Not determined by Lie(G), but involve 'global' considerations
- Typically finite order anomalies

[8] S. L. Adler, 1969. J. S. Bell and R. Jackiw, 1969.
[9] E. Witten, 1982.

Global anomalies in general?

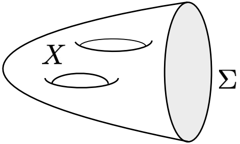
How can we systematically study global anomalies, if they can't be seen perturbatively? We need a better understanding of the object $Z_\psi[A,\Sigma] = \det i\not{D}$.

First observation:

$$Z_\psi[A,\Sigma] = \underbrace{|Z_\psi|}_{\text{anomaly free}} \; e^{i\theta}[A,\Sigma] \tag{7}$$

So the anomaly comes from the phase of the partition function. This phase can be understood using anomaly inflow.

Anomaly inflow: a simple example



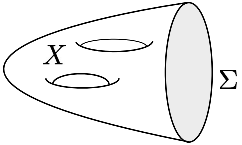
4d $U(1)$ gauge theory with a single Weyl fermion of unit charge. Under $\psi \to e^{i\alpha(x)}\psi$, $A \to A + d\alpha$,

$$Z_\psi \to \exp\left[-\frac{i}{8\pi^2}\int_\Sigma \alpha F \wedge F\right] Z_\psi$$

Anomaly reproduced by coupling to a classical 5d Chern–Simons term,

$$S_{\mathrm{CS}} = \frac{1}{8\pi^2}\int_X A \wedge F \wedge F; \quad \delta_\alpha S_{\mathrm{CS}} = \frac{1}{8\pi^2}\int_X d\left(\alpha F \wedge F\right) = \frac{1}{8\pi^2}\int_\Sigma \alpha F \wedge F$$

Anomaly inflow: (general) perturbative version



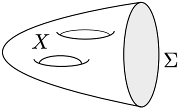
Whenever $\Sigma = \partial X$, with spin structure & map to BG extending to 5-mfd X, can reproduce perturbative anomaly with a 5d Chern–Simons term:

$$\underbrace{Z_\psi[A, \Sigma]}_{\text{4d partition fn}} = |Z_\psi| \exp\left(-2\pi i \int_X I_5\right) \tag{8}$$

Locally, dI_5 is the gauge-invariant 'anomaly polynomial':

$$dI_5 = \Phi_6 = \hat{A}(R)\ \mathrm{tr}\ \exp\left(\frac{iF}{2\pi}\right)\Big|_6 . \tag{9}$$

Anomaly inflow: non-perturbative version



Non-perturbative generalisation, still for $\Sigma = \partial X$, is[10]

$$Z_\psi[A, \Sigma] = |Z_\psi| \exp\left(-2\pi i \eta_X\right), \tag{10}$$

where η-invariant is regularised sum over eigenvalues λ_k of $i\not{D}_X$, e.g.

$$\eta_X = \lim_{\epsilon \to 0^+} \sum_k e^{-\epsilon|\lambda_k|} \text{sign}(\lambda_k)/2, \tag{11}$$

[10] E. Witten & K. Yonekura, 2019. See also E. Witten, 2015

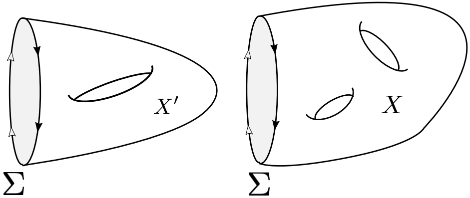
$Z_\psi = |Z_\psi| \exp\left(-2\pi i \eta_X\right)$ provides a suitable (smoothly-varying[11]) object for systematically studying local and global anomalies.

[11] X.-z. Dai and D. S. Freed, 1994

Anomalies from locality

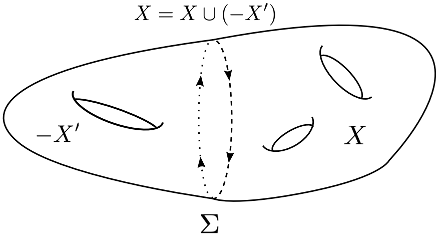
$$\boxed{Z_\psi[A,\Sigma] = |Z_\psi| \exp\left(-2\pi i \eta_X\right)} \tag{12}$$

A local 4d theory should be independent of the choice of 5d bulk X



$$\implies \exp\left(-2\pi i \eta'_X\right) = \exp\left(-2\pi i \eta_X\right) \tag{13}$$

Anomalies from locality



Use "gluing" property of η[12]

$$\exp\left(-2\pi i \eta'_X\right) = \exp\left(-2\pi i \eta_X\right) \implies \boxed{\exp\left(-2\pi i \eta_{\bar{X}}\right) = 1} \qquad (14)$$

Must hold for *any* closed 5-mfd $\bar{X}$ (that admits a spin structure and a map to BG). This condition will have very strong implications for anomalies

[12] X.-z. Dai and D. S. Freed, 1994

What is the connection to bordism?

Bordism and the η-invariant

Atiyah–Patodi–Singer (APS) index theorem for 6-mfd Y whose boundary $\bar{X} = \partial Y$ is a closed 5-mfd:[13]

$$\mathrm{Ind}(D_Y) = \int_Y \Phi_6 \underbrace{-\eta_{\bar{X}}}_{\text{'boundary correction'}} \qquad \text{(APS)}$$

Local anomalies: For $\bar{X} = \partial Y$, $(\eta_{\bar{X}} = \int_Y \Phi_6 = \int_{\bar{X}} I_5)$ mod $\mathbb{Z}$; reduces to perturbative anomaly inflow formula (Chern–Simons)

Cobordism invariance: When $\Phi_6 = 0$, $\eta_{\bar{X}} \in \mathbb{Z} \implies \exp(2\pi i \eta_{\bar{X}}) = 1$ for all $\bar{X} = \partial Y$ in the trivial bordism class.

$\implies \exp(2\pi i \eta_{\bar{X}})$ is a 5d (co)bordism invariant when $\Phi_6 = 0$[14]

[13] M. F. Atiyah, V. K. Patodi, and I. M. Singer, 1975.

[14] E. Witten, 1985. See also E. Witten, 2015.

A bordism criterion for global anomalies

Recall locality $\implies$ $\exp(2\pi i \eta_{\bar{X}}) = 1$ on all closed 5-mfds:

1. Considering mfds in trivial bordism class, already requires $\Phi_6 = 0$ (*i.e.* locality implies no perturbative anomalies)
2. If $\Phi_6 = 0$, may still be issues with locality on non-zero bordism classes. Would need to compute $\exp(2\pi i \eta_{\bar{X}})$ on suitable generators - hard in practice!
3. Cheat: $\exp(2\pi i \eta_{\bar{X}}) = 1$ necessarily holds on all closed 5-mfds if[15]

$$\boxed{\Omega_5^{\text{Spin}}(BG) = 0} \tag{15}$$

Then (a) the theory is local, and (b) the phase $\exp\left(-2\pi i \eta_{\bar{X}}\right)$ is trivial on any 'generalised mapping torus' $\bar{X}$, so no global anomalies.

This will be our (strong) criterion for there being no global anomaly.

[15] but not only if!

An important caveat

This whole analysis requires $\Sigma = \partial X$ for 5-mfd X with various structures extended.

But, generally, $\Omega_4^{\text{Spin}}(\cdot) \neq 0$, e.g. K3 surface.

Nonetheless, partition function can be consistently defined on all 4-mfds by assigning arbitrary theta angles to each generator of Ω_4.[16]

Theory is well-defined, but ambiguous.

[16] In string theory context, this is sometimes known as "setting the quantum integrand". See E. Witten, 1997 and D. Freed and G. Moore, 2006.

Another caveat

Q: What if spacetime itself has a boundary?

A: forget about it! (as far as I'm aware...)

Global anomalies in the SM(s)

Q: How do we compute $\Omega_5^{\text{Spin}}(BG)$, say for $G = SU(3) \times SU(2) \times U(1)$?

Bordism groups can often be computed using standard methods in algebraic topology.

Our tool of choice is the Atiyah–Hirzebruch spectral sequence.[17] We will here treat the AHSS as something of a black box, and only discuss what goes in, and what comes out.

[17] M. F. Atiyah and F. Hirzebruch, 1961.

Atiyah–Hirzebruch spectral sequence (AHSS)

- Spectral sequences are a kind of generalisation of exact sequences
- AHSS computes bordism groups of X where $F \to X \to B$
- For trivial fibration $\mathrm{pt} \to BG \to BG$,[18] inputs to the AHSS are

$$E_{p,q} := H_p(BG; \Omega_q^{\mathrm{Spin}}(\mathrm{pt})) = \underbrace{H_p(BG;\mathbb{Z})}_{\text{first input}} \otimes \underbrace{\Omega_q^{\mathrm{Spin}}(\mathrm{pt})}_{\text{second input}} \tag{16}$$

1. Build up homology from simpler spaces using $B(K \times H) = BK \times BH$ and Künneth theorem. E.g. $BU(1) = \mathbb{C}P^\infty$, $BSU(2) = \mathbb{H}P^\infty$.
2. Spin-bordism groups of a point are known:[19]

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$

(17)

[18] For G_{SM}/Γ_6 used alternative fibration $\mathbb{Z}/3 \longrightarrow U(2) \times SU(3) \longrightarrow G_{\mathrm{SM}}/\Gamma_6$

[19] D. Anderson, E. Brown Jnr, F. P. Peterson, 1966.

Our results for the SMs

G	$\Omega_d^{\mathrm{Spin}}(BG)$: 0	1	2	3	4	5
G_{SM}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2$
G_{SM}/Γ_2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}^4$	0
G_{SM}/Γ_3	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2$
G_{SM}/Γ_6	$\mathbb{Z}$	$\mathbb{Z}_2$	$\underbrace{e(\mathbb{Z}_3,\mathbb{Z}\times\mathbb{Z}_2)}_{\text{group extension}}$	0	$e(\mathbb{Z}_3,e(\mathbb{Z}_3,\mathbb{Z}^4))$	0

- First two columns only sensitive to spin structure
- Ω_d mostly boring in odd d ('opposite' situation to local anomalies)
- In all cases Ω_5 is 'at most' $\mathbb{Z}_2$ –
  no new global anomalies beyond the Witten anomaly.[20]

[20] See also Z. Wan & J. Wang, 1910.14668, which confirmed these results (and filled in the gaps) using the Adams spectral sequence.

Results for global anomalies in BSM gauge theories

No global anomalies (beyond Witten $SU(2)$ anomaly) in (multiple) Z' models, Pati-Salam unified theory, trinification models, or SM with a spin$_c$ structure (e.g. by gauging $B-L$)

G	$\Omega_d^{\mathrm{Spin}}(BG)$ 0	1	2	3	4	5
$U(1)^m \times SU(2) \times SU(3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^m \times \mathbb{Z}_2$	0	$\mathbb{Z}^{3+\frac{1}{2}m(m+1)}$	$\mathbb{Z}_2$
$SU(4) \times SU(2)_L \times SU(2)_R$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2^2$
$SU(3)_C \times SU(3)_L \times SU(3)_R$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^4$	0
$\frac{SU(3)_C \times SU(3)_L \times SU(3)_R}{\mathbb{Z}_3}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	0	$\mathbb{Z}^4$ or $\mathbb{Z}^4 \times \mathbb{Z}_3$	0
SM with spin$_c$structure	$\mathbb{Z}$	0	$\times$	0	$\times$	0

Lesson for model-builders: Global anomalies seem to be rather rare in BSM[21] – some reassurance for model builders!

[21] Though potentially more interesting in extra-dimension models...

Back to the SMs

G	$\Omega_d^{\text{Spin}}(BG)$ 0	1	2	3	4	5
G_{SM}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2$
G_{SM}/Γ_2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	0	$\mathbb{Z}^4$	0
G_{SM}/Γ_3	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2$
G_{SM}/Γ_6	$\mathbb{Z}$	$\mathbb{Z}_2$	$e(\mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_2)$	0	$e(\mathbb{Z}_3, e(\mathbb{Z}_3, \mathbb{Z}^4))$	0

In these two cases, there can be no global anomalies at all.

Physics Q: what happened to the Witten $SU(2)$ anomaly?

Anomaly Interplay in the SM

First, let's review the Witten $SU(2)$ anomaly again – but without mentioning $\pi_4(SU(2))\dots$

Recap: the $SU(2)$ anomaly

For single isospin-j fermion coupled to $SU(2)$ background F, Atiyah–Singer index theorem implies

$$\mathrm{Ind}(i\not{D}) := n_+ - n_- = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}\, F \wedge F = T(j)\; p_1(F), \tag{18}$$

where $p_1(F) \in \mathbb{Z}$ is instanton number, $T(j) = \frac{2}{3}j(j+1)(2j+1)$ is Dynkin index. Hence # of fermion zero modes (for p_1 odd) is

$$\mathcal{N}_j := n_+ + n_- \equiv T(j) \pmod 2. \tag{19}$$

If $\mathcal{N}_j$ odd, $Z[A]$ change signs under $(-1)^F$. But $(-1)^F$ equivalent to the gauge transformation $-\mathbf{1} \in SU(2)$, so $SU(2)$ is anomalous.

Recap: the $SU(2)$ anomaly

$$Z[A] \xrightarrow{-\mathbf{1}\in SU(2)} (-1)^{T(j)} Z[A] \tag{20}$$

$T(j) = \frac{2}{3}j(j+1)(2j+1)$ odd iff isospin $j = 2r + 1/2$; only these isospins contribute to this mod 2 anomaly.

Anomaly cancels iff an even number of fermions with isospins $2r + 1/2$.

Q: Why does $\Omega_5^{\text{Spin}}(BG_{\text{SM}}/\Gamma_{2,6}) \cong 0$? What has happened to the global $SU(2)$ anomaly?

The $SU(3)$ factor is here unimportant; can focus only on

$$\begin{array}{lcl} SU(2) \times U(1) & \text{vs.} & (SU(2) \times U(1))/\Gamma_2 \cong U(2) \\ \Omega_5^{\text{Spin}}(B\cdot) = \mathbb{Z}_2 & & \Omega_5^{\text{Spin}}(B\cdot) = 0 \end{array} \tag{21}$$

A: the global anomaly $SU(2)$ is traded for a local anomaly in $U(2)$.[22]

[22] JD and N. Lohitsiri, 2001.07731

Can see this in 3 ways. First we need some $U(2)$ rep theory:

$U(2)$ irreps labelled by an irrep of $SU(2)$ (isospin j) and a $U(1)$ charge q, such that

$$q \equiv 2j \pmod 2, \tag{22}$$

= an 'isospin-charge relation'.

[In general, $U(N)$ irreps labelled by an $SU(N)$ irrep and a $U(1)$ charge q satisfying

$$q = N\text{-ality} \tag{23}$$

of the $SU(N)$ rep.]

Method 1: the quick way

Mixed triangle anomaly is proportional to

$$\mathcal{A}_{\text{mix}} \equiv \sum_j T(j) \sum_{\alpha=1}^{N_j} q_{j,\alpha} = 0, \tag{24}$$

$T(j)$ is odd only for $j \in 2\mathbb{Z}_{\geq 0} + 1/2$, and $q \equiv 2j \pmod 2$. Hence, reducing mod 2:

$$\sum_{j \in 2\mathbb{Z}+1/2} 1 \equiv 0 \pmod 2, \tag{25}$$

so can be no Witten anomaly. But was this a coincidence?

Method 2: the physics way

In $U(2)$,

$$(-\mathbf{1}, 1) \sim (\mathbf{1}, e^{i\pi}) \in SU(2) \times U(1) \tag{26}$$

So the $SU(2)$ 'global gauge transformation' by $-\mathbf{1} \sim (-1)^F$ is actually a local $U(1)$ gauge transformation in $U(2)$.

Consider single $U(2)$ fermion with isospin j and charge q.
For $U(1)$ g. t. by angle θ, non-invariance of fermion measure gives

$$\begin{aligned} Z[A] &\to \exp\left[-\frac{iq\theta}{8\pi^2}\int_{S^4} \text{Tr } F \wedge F + \text{gravitational piece}\right] Z[A] \\ &= \exp\left[-iq\theta \;\; T(j) \;\; p_1(F)\right] Z[A], \\ &\xrightarrow{\theta=\pi,\; p_1 \text{odd}} (-1)^{qT(j)} Z[A] \end{aligned} \tag{27}$$

Non-anomalous iff an even number of fermions with $j = 2r + 1/2$. But this is just a perturbative anomaly, not a global anomaly.

Method 3: the maths way

Because $\Omega_5^{\mathrm{Spin}}(BU(2)) = 0$, can compute η-invariant directly, by using APS index theorem for any closed 5-mfd X:

$$\mathrm{ind}\left(i\not{D}\right) = \int_Y \Phi_6 - \eta_X. \tag{28}$$

On $X = M \times S^1$ mapping torus with $SU(2)$ 1-instanton through M, can extend $U(2)$ bundle to $Y = M \times D^2$, and evaluate

$$\begin{aligned}\exp(2\pi i \eta_X) &= \exp\left(2\pi i \int_{M\times D^2}\left[\frac{1}{24}p_1(\mathcal{R})\mathrm{Tr}\,\frac{\mathcal{F}}{2\pi} + \frac{1}{3!}\mathrm{Tr}\left(\frac{\mathcal{F}}{2\pi}\right)^3\right]\right) \\ &= \cdots = (-1)^{qT(j)}\end{aligned} \tag{29}$$

Unless "Witten condition" satisfied, partition function flips sign upon traversing mapping torus. A local anomaly because captured by Φ_6.

A more subtle anomaly interplay

We found a more subtle anomaly interplay occurs in $U(2)$ gauge theory defined without a spin-structure, involving both the 'old' and 'new'[23] $SU(2)$ anomalies – see back-up slides if interested!

[23] J. Wang, X-G. Wen, E. Witten, 2018.

Summary

- Non-perturbative anomaly inflow described by the η-invariant; possible global anomalies therefore detected by bordism groups
- We applied this criterion to the four SM gauge groups; found there is *at most* the $SU(2)$ Witten anomaly (same for several BSM theories)
- In two cases, there are *no global anomalies whatsoever*, due to 'anomaly interplay'
- P.S. more subtle interplay in non-spin $U(2)$ gauge theory

Visual summary:

	Local anomalies	Global anomalies
Even dimensions	Chern–Simons in $d+1$	Rare *e.g.* Witten $SU(2)$
Odd dimensions	Never!	Seemingly less rare ...

Thanks!

Postscript: $U(2)$ gauge theory without a spin structure

Recap: the 'new $SU(2)$ anomaly'

Can define an $SU(2)$ gauge theory without a spin structure (& $\therefore$ on non-spin mfds e.g. $\mathbb{C}P^2$), by using a 'spin-$SU(2)$ structure',

$$\text{Spin}_{SU(2)}(4) \equiv \frac{\text{Spin}(4) \times SU(2)}{\mathbb{Z}_2}, \tag{30}$$

if all fermions (bosons) have half-integral (integral) isospin.

Choose a spin-$SU(2)$ connection $A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, for a spin$_c$ connection a that obeys

$$\int_{\mathbb{C}P^1 \subset \mathbb{C}P^2} \frac{da}{2\pi} = \frac{1}{2} \tag{31}$$

Recap: the 'new $SU(2)$ anomaly'

The anomaly occurs only on certain non-spin mfds – let's take $M = \mathbb{C}P^2$, complex coords z_i. The anomaly is in the combination of a diffeomorphism plus gauge transformation, e.g.

$$\hat{\varphi} = \left\{ \begin{matrix} \varphi : z_i \mapsto z_i^* & \text{diffeo.} \\ W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SU(2), & \text{g.t.} \end{matrix} \right\} \quad (32)$$

which leaves the spin-$SU(2)$ connection A invariant.

Atiyah–Singer implies # fermion ZMs is

$$\mathfrak{I}_j = \mathcal{N}_j = \frac{1}{24}(4j^2 - 1)(2j + 3), \quad (33)$$

and they come in pairs with eigenvalues $+1$ and -1 under $\hat{\varphi}$. Hence

$$Z[A] \xrightarrow{\hat{\varphi}} (-1)^{\mathfrak{I}_j/2} Z[A]. \quad (34)$$

Recap: the 'new $SU(2)$ anomaly'

$$Z[A] \xrightarrow{\hat{\varphi}} (-1)^{\mathfrak{J}_j/2} Z[A], \tag{35}$$

$$\mathfrak{J}_j = \frac{1}{24}(4j^2 - 1)(2j + 3). \tag{36}$$

$\mathfrak{J}_j$ even for all half-integer j, but congruent to 2 mod 4 only when $j = 4r + 3/2$; only these isospins contribute to the new (mod 2) anomaly.

Anomaly cancels iff an even number of fermions with isospins $4r + 3/2$.

Now embed $SU(2) \to U(2)$

Define fields with a spin-$U(2)$ structure,

$$\mathrm{Spin}_{U(2)} \equiv \frac{\mathrm{Spin}(4) \times U(2)}{\mathbb{Z}/2}, \tag{37}$$

which requires

$$\begin{aligned} \text{fermion} &\longleftrightarrow j \in (2\mathbb{Z}+1)/2 &&\longleftrightarrow q \text{ odd}, \\ \text{boson} &\longleftrightarrow \quad j \in \mathbb{Z} &&\longleftrightarrow q \text{ even}. \end{aligned} \tag{38}$$

We consider a spin-$U(2)$ connection of the same form, $A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$. As for $SU(2)$ case, this theory can be put on any orientable 4-mfd.

Unlike the 'old' $SU(2)$ anomaly, the anomalous transformation $\hat{\varphi}$ is **not** equivalent to a local gauge transformation in $U(2)$.

But, at level of its action on $Z[A]$, it **is** equivalent to a local g.t. by

$$\tilde{W} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \in U(2). \tag{39}$$

Under $U(1)$ transformation by $e^{i\pi/2}$,

$$Z[A] \xrightarrow{\tilde{W}} Z[A] \exp \left[iS_{\text{gauge}} + \underbrace{iS_{\text{grav}}}_{\text{Non-vanishing on } \mathbb{C}P^2} \right], \tag{40}$$

$$S_{\text{gauge}} = -\frac{iq}{32\pi} \int_M \text{Tr } F_{\mu\nu} \tilde{F}^{\mu\nu} d^4x = -iq\pi T(j) \underbrace{\frac{1}{2} \int_M \frac{f \wedge f}{(2\pi)^2}}_{\frac{1}{8}\sigma}, \tag{41}$$

$$S_{\text{grav}} = iq\pi \frac{(2j+1)}{24} \underbrace{\frac{1}{2} \int_M \frac{\text{Tr } R \wedge R}{(2\pi)^2}}_{3\sigma}. \tag{42}$$

On $\mathbb{C}P^2$, signature $\sigma = 1$.

Hence

$$Z[A] \to Z[A] \exp \underbrace{\left[-\frac{i\pi}{8} \left(T(j) - \frac{1}{2}(2j+1) \right) q \right]}_{-i\pi \mathfrak{J}_j q/2}, \qquad (43)$$

thus

$$Z[A] \xrightarrow{\tilde{W}(\pi/2)} (-1)^{\mathfrak{J}_j q/2} Z[A]. \qquad (44)$$

Thus, we reproduce the condition for cancelling the new $SU(2)$ anomaly from a local $U(1)$ gauge transformation in $U(2)$. More mundanely, implied by taking a particular linear combination of anomaly coefficients,

$$\frac{1}{4}\left[\mathcal{A}_{\text{mix}} - \frac{1}{2}\mathcal{A}_{\text{grav}} \right] = \sum_{j \text{ half integer}} \mathfrak{J}_j \sum_{\alpha} q_{j,\alpha} = 0 \pmod 4 \qquad (45)$$

There is no possible 'new $U(2)$ anomaly', but by a sort of 'coincidence'. This statement can be better understood using cobordism.

Cobordism and the 'new' $U(2)$ anomaly

Firstly, for $SU(2)$ with spin-$SU(2)$ structure, both the 'old' and 'new' global anomalies captured by[24]

$$\Omega_5^{\frac{\text{Spin}\times SU(2)}{\mathbb{Z}/2}} = \mathbb{Z}/2 \times \mathbb{Z}/2 \qquad (46)$$

Possible basis for *co*bordism given by $\mathcal{I}_{1/2}$ and $\mathcal{I}_{3/2}$, the 5d mod 2 indices for single fermion with isospin-$1/2$ or $3/2$.

[24] J. Wang, X-G. Wen, E. Witten, 2018.

Cobordism and the 'new' $U(2)$ anomaly

For $U(2)$ with spin-$U(2)$ structure, we calculate using the Adams sequence that

$$\Omega_5^{\frac{\text{Spin}\times U(2)}{\mathbb{Z}/2}} = \mathbb{Z}/2 \tag{47}$$

No 'old' $U(2)$ anomaly corresponding to $\mathcal{I}_{1/2}$. But the 'new' anomaly 'still there', detected by

$$\int_X w_2 w_3, \tag{48}$$

which is actually a cobordism invariant independent of the $U(2)$-structure.

Disentangling the anomaly interplay

In what sense is the new $U(2)$ global anomaly 'still there' physically, beyond the result of the bordism calculation?

A low-energy theory with this anomaly can be revealed by cancelling the perturbative anomalies using Wess–Zumino terms,

$$\mathcal{L} \to \mathcal{L} + \frac{i\mathcal{A}_{\text{mix}}}{32\pi^2}\phi F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \frac{i\mathcal{A}_{\text{grav}}}{384\pi^2}\phi\sqrt{g}R_{\mu\nu\sigma\tau}\tilde{R}^{\mu\nu\sigma\tau}, \tag{49}$$

albeit at the expense of spontaneously breaking $U(2) \to SU(2)$...

The new $(S)U(2)$ anomaly that remains can then be cancelled by coupling to a TQFT.[25]

[25] Kapustin, 2014. Thorngren, 2014.

We could summarize this story as follows:

> It is possible to write down a consistent $U(2)$ theory of a single isospin-$3/2$ fermion, that can be defined on non-spin manifolds using a spin-$U(2)$ structure, if one includes a pair of WZ terms to cancel the perturbative anomalies, and couples to a tQFT to cancel the residual global anomaly.