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General Relativity
Sheet 9

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Prof. P. Jetzer

M. Haney, S. Tiwari, A. Boitier, M. Ebersold, A. Dei, L. Fritz
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Exercise 1 [Curvature for extended metrics]

Starting from an n-dimensional metric h with $ds^2_{(n)} = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$, one can define an $(n+1)$-dimensional metric g with

$$ds^2_{(n+1)} = \epsilon \mathrm{d}y^2 + f(y)h_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j,$$

with $\epsilon = \pm 1$.

a) Show that the only non-vanishing Christoffel symbols of g are:

$$\Gamma^y{}_{ij} = -\frac{\epsilon}{2}f' h_{ij}, \qquad \Gamma^i{}_{jy} = \frac{f'}{2f}\delta^i{}_j, \qquad \Gamma^k{}_{ij} = {}^{(n)}\Gamma^k{}_{ij}(h),$$

where ${}^{(n)}\Gamma^k{}_{ij}(h)$ is the n-dimensional Christoffel symbol of the metric h, and prime denotes derivation with respect to y.

b) Show that the Riemann tensor of g is given by:

$$R^i{}_{yjy} = -\left(\frac{f''}{2f} - \frac{f'^2}{4f^2}\right)\delta^i{}_j, \qquad R^y{}_{iyj} = -\epsilon\left(\frac{f''}{2} - \frac{f'^2}{4f}\right)h_{ij},$$

$$R^l{}_{ijk} = {}^{(n)}R^l{}_{ijk} - \epsilon\frac{f'^2}{4f}\left(h_{ik}\delta^l{}_j - h_{ij}\delta^l{}_k\right).$$

Using these relations one can obtain, by recursion, the components of the Riemann tensor of several useful metrics.

In the following, compute only Riemann tensors of the form $R^\alpha{}_{\beta\alpha\beta}$, since by symmetry the Ricci tensor will be diagonal in part e).

c) Starting from $h_{(1)} = \mathrm{d}\phi^2$, compute the Riemann tensor of

$$g_{(2)} = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$$

d) Starting from $g_{(2)}$, compute the Riemann tensor of

$$g_{(3)} = \mathrm{d}\chi^2 + f^2(\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

e) Using these results, compute the Ricci tensor and scalar for the metric

$$g_{(4)} = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + f^2(\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right],$$

where

$$f(\chi) = \begin{cases} \sin\chi & \text{or} \\ \chi & \text{or} \\ \sinh\chi \end{cases}$$

Exercise 2 [Geodesic separation]

The defining property of Euclidean (flat) geometry is that initially parallel lines remain parallel forever. This no longer holds on curved manifolds. Here, we attempt to quantify this behavior for an arbitrary space-time.

Consider a pair of freely falling test particles traveling on trajectories $x^\mu(\lambda)$ and $x^\mu(\lambda) + \delta x^\mu(\lambda)$. Show that the deviation between the two is given by solutions to

$$\frac{D^2}{D\lambda^2}\delta x^\mu = R^\mu{}_{\nu\sigma\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda}\delta x^\rho, \tag{1}$$

where the covariant derivatives here are taken along the curve $x^\mu(\lambda)$.