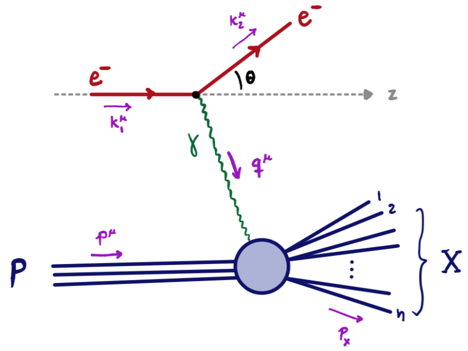
Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 8

Exercise 1: Quark spin and DIS

The Deep inelastic scattering (DIS) $e^- p \to e^- X$ is a QED process described by the following kinematic configuration



where X denotes any number of final state hadrons with total momentum $p_X = p + q$. The momentum transfer is defined as $Q^2 = -q^2$ and the momentum fraction of the proton is given by the Björken variable $x = Q^2/(2p \cdot q)$. In the previous exercise sheet, we have shown that the DIS cross-section can be brought to the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2 y^2}{Q^6}[k_1^\mu k_2^\nu + k_2^\mu k_1^\nu - (k_1 \cdot k_2) g^{\mu\nu}] W_{\mu\nu} \,, \tag{1}$$

where $\alpha = e^2/4\pi$ is the fine-structure constant, $y = (p \cdot q)/(p \cdot k_1)$ is a kinematical variable and $W^{\mu\nu}$ is a purely hadronic tensor, which takes the form

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \left(p^\mu - \frac{p \cdot q}{q^2}\, q^\mu\right)\left(p^\nu - \frac{p \cdot q}{q^2}\, q^\nu\right)\frac{F_2}{p \cdot q} \,, \tag{2}$$

where $F_{1,2} \equiv F_{1,2}(x, Q^2)$ are the so-called structure functions. For an elastic process $eq \to eq$, we have also shown that the tensor $W_{\mu\nu}$ reads

$$W^{\mu\nu}\Big|_{\text{elastic}} = L_{qq}^{\mu\nu}\, \frac{\hat{x}}{2Q^2}\, \delta(1 - \hat{x}) \,, \tag{3}$$

with

$$L_{qq}^{\mu\nu} = 2\, e_q^2 \left[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - (p_1 \cdot p_2) g^{\mu\nu}\right] \,, \tag{4}$$

where p_1 (p_2) is the momentum of the ingoing (outgoing) quark and e_q is the quark-q electric charge in units of e.

a) Replace Eq. (2) into Eq. (1) to show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{2\pi\alpha^2}{xQ^4}\Big[\left(1 + (1-y)^2\right) F_2(x, Q^2) - y^2 F_L(x, Q^2)\Big] \,, \tag{5}$$

where $F_L(x, Q^2) \equiv F_2(x, Q^2) - 2xF_1(x, Q^2)\,.$

b) Let us express the structure functions $F_i(x, Q^2)$ in terms of the partonic ones, as discussed in the lecture notes,

$$\frac{F_i(x,Q^2)}{x} = \sum_q \int_0^1 \frac{\mathrm{d}\xi}{\xi} f_q(\xi) \frac{\hat{F}_i(\hat{x},Q^2)}{\hat{x}}\bigg|_{\hat{x}=x/\xi} \tag{6}$$

where f_q denotes the parton-distribution function of the quark q. Derive explicitly the Callan-Gross relation,

$$\frac{F_2(x,Q^2)}{x} = 2\,F_1(x,Q^2) = \sum_q e_q^2\, f_q(x)\,. \tag{7}$$

c) Let us now assume that quarks ϕ_q are spin-0 particles. In this case, their interactions with photons would be described by the following Lagrangian,

$$\mathcal{L} \supset -i\, e\, e_q\, A_\mu \left[\phi_q^*(\partial_\mu \phi_q) - (\partial_\mu \phi_q)^* \phi_q\right] + e^2\, e_q^2\, A_\mu A^\mu |\,\phi_q|^2\,. \tag{8}$$

What would be the F_1 and F_2 expressions in Eq. (7) in this case?

Exercise 2: Neutrino-nucleon DIS

The charged-current interactions between quark and leptons is described in the Fermi Theory by the following Lagrangian,

$$\mathcal{L}_{\text{Fermi}} \supset -\frac{G_{\text{F}}}{\sqrt{2}}\Big[(\bar{u}\gamma^\mu(1-\gamma_5)d)\left(\bar{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell\right) + \left(\bar{d}\gamma^\mu(1-\gamma_5)u\right)(\bar{\nu}_\ell\gamma_\mu(1-\gamma_5)\ell)\Big]\,, \tag{9}$$

where $G_F \approx 1.16 \times 10^{-5}$ GeV^{-2} is the fermi constant, ℓ denotes a charged lepton $(e, \mu\,\tau)$, and u and d stand for the up- and down-quarks, respectively.

a) Show that the squared matrix elements for $\nu_\mu d \to \mu^- u$ and $\nu_\mu \bar{u} \to \mu^- \bar{d}$ read

$$\overline{\sum}|\mathcal{M}(\nu_\mu d \to \mu^- u)|^2 = 16 G_F^2 \hat{s}^2\,, \tag{10}$$

$$\overline{\sum}|\mathcal{M}(\nu_\mu \bar{u} \to \mu^- \bar{d})|^2 = 16 G_F^2 \hat{u}^2 \tag{11}$$

where $\sqrt{\hat{s}}$ is partonic center-of-mass energy, and where we have assumed $\sqrt{\hat{s}} \gg m_\mu$. Derive the total cross-sections for these processes.

b) Show that the neutrino-nucleon DIS cross-sections can be expressed in terms of the quark PDFs as follows,

$$\sigma(\nu_\mu N \to \mu^- X) = \frac{G_F^2 s}{\pi} \int_0^1 \mathrm{d}x\, x \left[f_d^{(N)}(x) + \frac{1}{3} f_{\bar{u}}^{(N)}(x)\right]\,, \tag{12}$$

$$\sigma(\bar{\nu}_\mu N \to \mu^+ X) = \frac{G_F^2 s}{\pi} \int_0^1 \mathrm{d}x\, x \left[f_{\bar{d}}^{(N)}(x) + \frac{1}{3} f_u^{(N)}(x)\right]\,. \tag{13}$$

Summing over $N = n, p$ and using the isospin relations $f_u^{(p)} = f_d^{(n)}$ and $f_d^{(p)} = f_u^{(n)}$, show that

$$\begin{aligned} \sigma(\nu_\mu N \to \mu^- X) &= 3\,\sigma(\bar{\nu}_\mu N \to \mu^+ X) \\ &= \frac{G_F^2 s}{\pi} \int_0^1 \mathrm{d}x\, x \left[f_d^{(N)}(x) + f_u^{(N)}(x) \right] , \end{aligned} \tag{14}$$

where we have kept only the valence-quark contributions.

c) The Icecube experiment measures the scattering of high-energy neutrinos with the detector that is made of Antartic ice. The incoming (anti-)neutrinos can have any flavor (e, μ, τ) and their energies can be as large as $E_\nu \simeq 10$ PeV. Let us consider the process $\nu_\mu N \to \mu^- X$ studied above. Do you expect Eq. (9) to provide a reasonable description of this process? What if the target is an electron instead of a nucleon?