Exercise 1. *One-loop vacuum polarisation in QED*

1. *In massive QED,*

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi - m\bar{\psi}\psi \,, \tag{1}$$

consider quantum corrections to the photon polarisation function $i\Pi^{\mu\nu}(q)$. Show that, at one-loop order, $\Pi^{\mu\nu}(q)$ receives contribution from a single diagram. Apply the Feynman rules to verify that the expression of the latter is

$$i\Pi_2^{\mu\nu}(p) = -e^2 \mu^{2\epsilon}\, \mathrm{Tr}[\mathbf{1}] \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N^{\mu\nu}}{(k^2 - m^2)((k+p)^2 - m^2)} \,, \tag{2}$$

with $\epsilon = (4-d)/2$ and $N^{\mu\nu} = 2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}(k^2 + k\cdot p - m^2)$.

2. *Show that, by symmetry, one has*

$$\begin{aligned} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu}{(k^2-\Delta)^a} &= 0 \,, \\ \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2-\Delta)^a} &= \frac{1}{d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{g^{\mu\nu} k^2}{(k^2-\Delta)^a} \,. \end{aligned} \tag{3}$$

3. *Use the Feynman parametrisation to show that*

$$i\Pi_2^{\mu\nu}(p) = -e^2 \mu^{2\epsilon}\, \mathrm{Tr}[\mathbf{1}] \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{N'^{\mu\nu}}{(k^2-\Delta)^2} \,, \tag{4}$$

with $\Delta = m^2 - x(1-x)p^2$ and

$$N'^{\mu\nu} = g^{\mu\nu} k^2 \left(\frac{2}{d} - 1\right) - 2x(1-x) p^\mu p^\nu + g^{\mu\nu}(x(1-x)p^2 + m^2) \,. \tag{5}$$

Finally, use the result of the Exercise Sheet 1 to integrate over the loop momentum and obtain

$$i\Pi_2^{\mu\nu}(p) = -i(p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi_2(p^2) \,, \tag{6}$$

with

$$\Pi_2(p^2) = \frac{2e^2\mu^{2\epsilon}}{(4\pi)^{d/2}} \mathrm{Tr}[\mathbf{1}]\, \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 \mathrm{d}x\, x(1-x) \Delta^{d/2-2} \,. \tag{7}$$

Argue that this result is consistent with gauge invariance.

In the case of massless electron, compute the integral over the Feynman parameter and get

$$\Pi_2(p^2)|_{m=0} = \frac{2e^2\mu^{2\epsilon}}{(4\pi)^{d/2}} \mathrm{Tr}[\mathbf{1}] (-p^2)^{d/2-2} \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} \,. \tag{8}$$

In the $m \neq 0$ case, expand $i\Pi_2(p^2)$ around $\epsilon = 0$ ($\mathrm{Tr}[\mathbf{1}] = 4$) and verify that the vacuum polarisation has a single pole in four dimensions, whose residue corresponds to the one of the massless calculation (S.15),

$$\Pi_2(q) = \frac{e^2}{2\pi^2} \int_0^1 \mathrm{d}x\, x(1-x) \left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi - \log \frac{\Delta}{\mu^2}\right) + \mathcal{O}(\epsilon) \,. \tag{9}$$

4. *In the renormalised QED Lagrangian,* $\mathcal{L}_R = \mathcal{L} + \mathcal{L}_{c.t.}$*,*

$$\mathcal{L}_{c.t.} = -\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\left(\delta_2 \not{\partial} + ie_R\delta_1 \not{A}\right)\psi + (\delta_2 + \delta_m)m_R\bar{\psi}\psi\,, \tag{10}$$

the photon wave function renormalisation constant δ_3 *introduces a new coupling,*

$$= -i\delta_3(p^2 g^{\mu\nu} - p^\mu p^\nu)\,, \tag{11}$$

which contributes to $i\Pi^{\mu\nu}$*. In the (modified) minimal subtraction scheme (*$\overline{MS}$*), the counterterms are chosen in order to remove, besides the UV divergent terms of Green's functions, also the finite terms proportional to* γ_E *and* $\log 4\pi$*, which are produced by dimensional regularisation. Compute* $\delta_3^{\overline{MS}}$*.*

5. *Alternatively, in the on-shell renormalisation scheme (OS),* δ_i *are fixed in such a way that parameters of the renormalised Lagrangian correspond to the physical ones. In the case of the photon wave function, this amounts to require that the renormalised propagator* $iG^{\mu\nu}$ *has a pole at* $p^2 = 0$ *with residue* $-i$*.*

   By starting from the bare photon propagator in Lorentz gauge,

$$iG^{\mu\nu}_{bare} = -i\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2}\,, \tag{12}$$

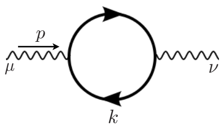
   compute the renormalised propagator $iG^{\mu\nu}$ *up to* $\mathcal{O}(e_R^4)$ *and determine* δ_3^{OS}*.*

   Verify that, conversely to the $\overline{MS}$ *scheme, in the on-shell scheme the renormalised propagator becomes independent of the subtraction point* μ*.*

Solution.

1. At one-loop level, one single diagram with a closed fermion loop contributes to $i\Pi_2^{\mu\nu}(p)$. By applying the QED Feynman rules we find

$$i\Pi_2^{\mu\nu}(p) =$$



$$= -e^2\mu^{2\epsilon}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{p}+m)\right]}{(k^2-m^2)((k+p)^2-m^2)}\,. \tag{S.1}$$

   The non-vanishing contributions to the numerator contain the trace of either two or four γ-matrices, which are given by

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[\mathbf{1}]\, g^{\mu\nu}\,,\\ \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= \mathrm{Tr}[\mathbf{1}]\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,.\end{aligned} \tag{S.2}$$

   By inserting these results into eq. (S.1), we immediately obtain eq. (2).

2. The rank-1 tensor integral vanishes by parity, since it corresponds to the integral over a symmetric domain of an odd function (in the μ-component of the loop momentum). For a similar reason, the rank-2 integral vanishes unless $\mu = \nu$. Hence, it must be proportional to $g^{\mu\nu}$,

$$\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2-\Delta)^a} = g^{\mu\nu}A(\Delta)\,. \tag{S.3}$$

   The coefficient $A(\Delta)$ can be determined by contracting both sides of the previous equation with $g_{\mu\nu}$,

$$\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^2}{(k^2-\Delta)^a} = d\,A(\Delta) \quad\Rightarrow\quad A(\Delta) = \frac{1}{d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^2}{(k^2-\Delta)^a}\,, \tag{S.4}$$

   which correctly gives eq. (3).

3. We combine the two denominators of eq. (2) by introducing the Feynman parameter x,

$$i\Pi_2^{\mu\nu}(p) = -e^2\mu^{2\epsilon}\,\mathrm{Tr}[\mathbf{1}]\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{N^{\mu\nu}}{[(1-x)(k^2-m^2)+x((k+p)^2-m^2)]^2}\,, \tag{S.5}$$

and then we complete the square through the shift of loop momentum $k'^{\mu} = k^\mu + xp^\mu$,

$$i\Pi_2^{\mu\nu}(p) = -e^2\mu^{2\epsilon}\,\mathrm{Tr}[\mathbf{1}]\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{N'^{\mu\nu}}{(k^2-\Delta)^2}\,, \tag{S.6}$$

with $\Delta = m^2 - x(1-x)p^2$ and

$$N'^{\mu\nu} = 2k^\mu k^\nu - 2x(1-x)p^\mu p^\nu - g^{\mu\nu}(k^2 - x(1-x)p^2 - m^2)\,. \tag{S.7}$$

In $N'^{\mu\nu}$ all terms $\sim k^\mu$ have been neglected since, according to eq. (3), they would vanish after integration. In addition, eq. (3) can be used to substitute $k^\mu k^\nu \to g^{\mu\nu}/d$ and reduce $N'^{\mu\nu}$ to

$$N'^{\mu\nu} = g^{\mu\nu}k^2\left(\frac{2}{d}-1\right) - 2x(1-x)p^\mu p^\nu + g^{\mu\nu}(x(1-x)p^2 + m^2)\,. \tag{S.8}$$

We can now perform the Wick rotation $k^0 \to ik_E^0$ (note that $k^2 \to -k_E^2$) and compute the integrals

$$\begin{aligned}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta)^2} &= \frac{i}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\frac{d}{2}\Delta^{d/2-1}\,,\\ \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^2}{(k^2-\Delta)^2} &= -\frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-q}\,,\end{aligned} \tag{S.9}$$

which, once they have been inserted into eq. (S.11), give

$$\begin{aligned} i\Pi_2^{\mu\nu}(p) =& \frac{-ie^2\mu^{2\epsilon}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\mathrm{Tr}[\mathbf{1}]\int_0^1 \mathrm{d}x\,\left[g^{\mu\nu}(x(1-x)p^2+m^2-\Delta) - 2x(1-x)p^{\mu\nu}\right]\Delta^{d/2-2}\,,\\ =& \frac{-2ie^2\mu^{2\epsilon}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\mathrm{Tr}[\mathbf{1}](p^2 g^{\mu\nu} - p^\mu p^\nu)\int_0^1 \mathrm{d}x\,x(1-x)\Delta^{d/2-2}\,.\end{aligned} \tag{S.10}$$

Therefore, we can set

$$i\Pi_2^{\mu\nu}(p) = -i(p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi_2(p^2)\,, \tag{S.11}$$

with Π_2 defined as in eq. (7). The tensor structure of eq. (S.11) could have been guessed from gauge invariance. By Lorentz invariance, $i\Pi^{\mu\nu}(p)$ can be decomposed as

$$i\Pi^{\mu\nu}(p) = A(p^2)g^{\mu\nu} + B(p^2)p^\mu p^\nu\,, \tag{S.12}$$

with A and B being scalar functions of p^2. The Ward identity demands $i\Pi^{\mu\nu}(p)$ to vanish when it is contracted with the photon momentum,

$$i\Pi^{\mu\nu}(p)p_\mu = A(p^2)p^\nu + B(p^2)p^2p^\nu = 0\,, \quad \Rightarrow \quad A(p^2) = -p^2 B(p^2)\,. \tag{S.13}$$

Hence, $i\Pi^{\mu\nu}(p)$ depends on a single scalar function $B(p^2) \equiv i\Pi(p^2)$,

$$i\Pi_2^{\mu\nu}(p) = -i(p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p^2)\,. \tag{S.14}$$

Note that the above relation holds at any order in the perturbative expansion.

In the massless case, $\Delta|_{m=0} = -p^2\, x(1-x)$ and the integral over the Feynman parameter is reduced to the Beta-function,

$$
\begin{aligned}
\Pi_2(p^2)|_{m=0} =& \frac{2e^2\mu^{2\epsilon}}{(4\pi)^{d/2}}\,\mathrm{Tr}[\mathbf{1}](-p^2)^{d/2-2}\int_0^1 \mathrm{d}x\, x^{d/2-1}(1-x)^{d/2-1}\,,\\
=& \frac{2e^2\mu^{2\epsilon}}{(4\pi)^{d/2}}\,\mathrm{Tr}[\mathbf{1}](-p^2)^{d/2-2}\,\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}\,.
\end{aligned}
\tag{S.15}
$$

For a massive electron, we can expand eq. (7) around $\epsilon = 0$, by making use of

$$
\begin{aligned}
\Gamma(\epsilon) =& \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\,,\\
a^\epsilon =& 1 + \epsilon\log a + \mathcal{O}(\epsilon^2)\,,
\end{aligned}
\tag{S.16}
$$

which lead to

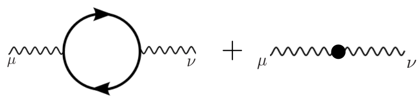
$$
\begin{aligned}
i\Pi_2(p^2) =& \frac{e^2}{2\pi^2}(1+\epsilon\log\mu^2+\cdots)(1+\epsilon\log(4\pi)+\cdots)\left(\frac{1}{\epsilon}-\gamma_E+\ldots\right)\times\\
&\int_0^1 \mathrm{d}x\,(1-\epsilon\log\Delta+\ldots)\,x(1-x)\\
=& \frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{1}{\epsilon}-\gamma_E+\log 4\pi-\log\frac{\Delta}{\mu^2}\right)+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{S.17}
$$

The UV divergent part of $i\Pi_2(p^2)$ is given by

$$
i\Pi_2^{\mathrm{UV}}(p^2) = \frac{e^2}{2\pi^2}\frac{1}{\epsilon}\int_0^1 \mathrm{d}x\, x(1-x) = \frac{e^2}{12\pi^2}\frac{1}{\epsilon}\,. \tag{S.18}
$$

$i\Pi_2^{\mathrm{UV}}(p^2)$ is independent of the electron mass and, hence, it is the same as $i\Pi_2^{\mathrm{UV}}(p^2)|_{m=0}$, as it can be verified from the series expansion of $i\Pi_2(p^2)|_{m=0}$.

4. In the renormalised Lagrangian, the one-loop contribution to the vacuum polarisation becomes

$i\overline{\Pi}^{\mu\nu} =$  $= -i(p^2 g^{\mu\nu} - p^\mu p^\nu)(\Pi_2(q^2)+\delta_3)\,.$ (S.19)

In the $\overline{\mathrm{MS}}$ scheme, the one-loop countertem $\delta_3^{\overline{\mathrm{MS}}}$ is fixed in such a way to remove $\Pi_2^{\mathrm{UV}}(p^2)$ as well as the finite terms induced by dimensional regularisation,

$$
\delta_3^{\overline{\mathrm{MS}}} = -\frac{e_R^2}{2\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\log 4\pi\right)\int_0^1 \mathrm{d}x\, x(1-x) = -\frac{e_R^2}{12\pi^2}\left(\frac{1}{\epsilon}+\log\left(4\pi e^{-\gamma_E}\right)\right)\,, \tag{S.20}
$$

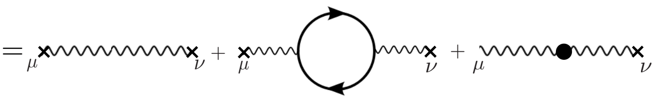
where we have identified e with the renormalised electron charge e_R, since the difference between the two would amount to higher order corrections.

5. We observe that, in Lorentz gauge, the bare photon propagator $iG^{\mu\nu}_{\mathrm{bare}}$ has the same tensor structure as $i\Pi^{\mu\nu}$ and that $P^{\mu\nu}(p) = (g^{\mu\nu} - p^\mu p^\nu/p^2)$ is a projector operator, since

$$
P^{\mu\alpha}P_\alpha^{\nu} = P^{\mu\nu}\,. \tag{S.21}
$$

Hence, the renormalised photon propagator is

$$-iG^{\mu\nu} =$$



$$+\,\mathcal{O}(e_R^4)$$

$$\begin{aligned}
&= -\,iG^{\mu\nu}_{\text{bare}} - iG^{\mu\alpha}_{\text{bare}}\left[-iP_{\alpha\beta}\,p^2\left(\Pi_2(p^2)+\delta_3\right)\right](-iG^{\beta\nu}_{\text{bare}}) + \mathcal{O}(e_R^4)\\
&= -\,iG^{\mu\nu}_{\text{bare}}\left(1-\Pi_2(p^2)-\delta_3\right) + \mathcal{O}(e_R^4)\\
&= -\,i\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\frac{1-\Pi_2(p^2)-\delta_3}{p^2} + \mathcal{O}(e_R^4)\,. \qquad \text{(S.22)}
\end{aligned}$$

We see that $-iG^{\mu\nu}$ has a pole at $p^2=0$. Therefore, the on-shell renormalisation constant can be fixed just by demanding that the residue of eq. (S.22) at the pole is equal to $-i$,

$$\begin{aligned}
\delta_3^{\text{OS}} \equiv -\Pi_2(0) &= -\,\frac{e_R^2}{2\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\log 4\pi+\log\frac{\mu^2}{m_R^2}\right)\int_0^1 \mathrm{d}x\,x(1-x)\\
&= -\,\frac{e_R^2}{12\pi^2}\left(\frac{1}{\epsilon}+\log\left(\frac{\mu^2 4\pi e^{-\gamma_E}}{m_R^2}\right)\right)\,. \qquad \text{(S.23)}
\end{aligned}$$

With this definition, the renormalised propagator becomes

$$-iG^{\mu\nu} = -i\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\left(1-\frac{e_R^2}{2\pi^2}\int_0^1 \mathrm{d}x\,x(1-x)\log\left(\frac{m_R^2}{\Delta}\right)\right)\,, \qquad \text{(S.24)}$$

which turns out to be independent of the subtraction point μ.