A Basis of Finite Feynman Integrals

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based on work with Erik Panzer and Robert M. Schabinger

1411.7392, 1510.06758

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Multi-loop Feynman integrals

consider L loop Euclidean Feynman integrals:

$$I = \left(\frac{\Gamma\left(\frac{d}{2} - 1\right)}{i\pi^{\frac{d}{2}}} \right)^L \int \mathrm{d}^d k_1 \cdots \int \mathrm{d}^d k_L \frac{1}{D_1^{a_1} \cdots D_N^{a_N}}$$

where $a_i \in \mathbb{Z}$ and e.g. $D_1 = k_1^2 - m_1^2$ etc.

linear dependencies:

- integration-by-parts (IBP) identities [Tkachov, Chetyrkin '81]
- systematic reduction to small number of master integrals [Laporta '00]
- think of it as linear vector space with some arbitrary basis (master integrals)

expansion in $\epsilon = (4 - d)/2$

- typically sufficient for phenomenological applications
- Laurent coefficients are simpler integrals

solving methods typically based on

1. direct integration of Feynman parameter integrals
2. differential equations

basis of integrals: choose according to integration method

An improved basis for Feynman parameters

consider Feynman parameter representation of multi-loop integral

$$I = \frac{\Gamma^L\left(\frac{d}{2}-1\right)\Gamma\left(\nu - L\frac{d}{2}\right)(-1)^\nu}{\prod_{i=1}^N \Gamma(\nu_i)} \left[\prod_{j=1}^N \int_0^\infty \mathrm{d}x_j x_j^{\nu_k - 1}\right] \delta(1 - x_N)\, \mathcal{U}^{\nu - (L+1)\frac{d}{2}} \mathcal{F}^{-\nu + L\frac{d}{2}}$$

where $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity

presence of subdivergencies (= divergencies from Feynman parameter integrations) implies:

- can't directly expand in ϵ
- no straight-forward analytical or numerical integration

generic approaches to singularity resolution:

1. sector decomposition [Hepp '66, Binoth, Heinrich '00]
2. polynomial exponent raising [Bernstein '72, Tkachov '96, Passarino '00]
3. analytic regularisation [Panzer '14]
4. New basis of finite Feynman integrals ("dims & dots") [AvM, Schabinger, Panzer '14]

Sector decomposition: shortcomings

calculate to $\mathcal{O}(\epsilon)$:

$$I(\epsilon) = \int_0^1 \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$:

$$I_1(\epsilon) = \int_0^{1/2} \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$
$$I_2(\epsilon) = \int_{1/2}^1 \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)\,.$$

rescale, expand in plus distributions, evaluate:

$$I_1(\epsilon) = -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3}\pi^2 - 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$
$$I_2(\epsilon) = -\frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3}\pi^2 + 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

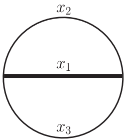
result:

$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3}\pi^2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

note:

- split up of domain introduces spurious terms ln(2)
- spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility & prevents analytical integration a la [Brown '08; Panzer '14]

An example for subdivergencies



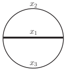
$$= \left(\frac{\Gamma(1-\epsilon))}{i\pi^{\frac{d}{2}}}\right)^2 \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \frac{1}{((k_1+k_2)^2 - m^2)\, k_1^2 k_2^2}$$

$$= -\Gamma^2(1-\epsilon)\Gamma(-1+2\epsilon) \int_0^\infty \mathrm{d}x_1 \delta(1-x_1) \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{1-2\epsilon}\,,$$

with Symanzik polynomials

$$\mathcal{U} = x_1x_2 + x_1x_3 + x_2x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U}\,.$$

- can't expand integrand in ϵ:



$$= -\left(m^2\right)^{1-2\epsilon} \frac{\Gamma(-1+2\epsilon)\Gamma(\epsilon)\Gamma^3(1-\epsilon)}{1-\epsilon}$$

$\Gamma(\epsilon)$ signals subdivergence

- Euclidean integrals: all divergencies from integration boundaries
- notation here: restrict to one or several parameters approaching zero (not infinity)

Systematic recognition of subdivergencies

- follow [Panzer '14]
- consider proper subsets

$$\{x_1, x_2\}, \quad \{x_1, x_3\}, \quad \{x_2, x_3\}, \quad \{x_1\}, \quad \{x_2\}, \quad \{x_3\}$$

- for each subset J consider scaling with λ:

$$J \to \lambda J$$

for integrand $P \equiv \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{1-2\epsilon}$:

$$P \to P_{J_\lambda} = \lambda^{\deg_J(P)} \tilde{P} \qquad \text{where} \quad \lim_{\lambda \to 0} \tilde{P} = \mathcal{O}(\lambda^0)$$

and the integral measure

$$\prod_{i=1}^{3} \mathrm{d}x_i \to \lambda^{|J|} \prod_{i=1}^{3} \mathrm{d}x_i$$

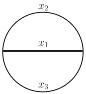
and read off:

Convergence index

$$\omega_J(P) = |J| + \deg_J(P)\,,$$

$$\lim_{\epsilon \to 0} \omega_J(P) \leq 0 \quad \Leftrightarrow \quad \text{presence of non-integrable subdivergence}$$

An example for subdivergencies: convergence index



$$= -\Gamma^2(1-\epsilon)\Gamma(-1+2\epsilon)\int_0^\infty \mathrm{d}x_1 \delta(1-x_1)\int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, \mathcal{U}^{-3+3\epsilon}\mathcal{F}^{1-2\epsilon}\,,$$

with Symanzik polynomials

$$\mathcal{U} = x_1x_2 + x_1x_3 + x_2x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U}\,.$$

for $J = \{x_2, x_3\}$:

$$P|_{J\to\lambda J} = \lambda^{\epsilon-2}\left(m^2x_1\right)^{1-2\epsilon}(x_1x_2 + x_1x_3 + \lambda x_2x_3)^{\epsilon-2}$$

and

$$\omega_{\{x_2,x_3\}}(P) = \epsilon$$

signals subdivergence

Analytic regularisation

integrand can be regularised by iterative procedure [Panzer '14]:

1. pick J for which $\lim_{\epsilon\to 0}\omega_J(P) \leq 0$
2. multiply by $1 = \int_0^\infty \mathrm{d}\lambda\ \delta(\lambda - x_J)$ with $x_J = \sum_{j\in J} x_j$
3. rescale $x_j \to \lambda x_j$ for all $j \in J$ and perform partial integration (surface term vanishes)
4. new integrand
$$P' = -\frac{1}{\omega_J(P)}\frac{\partial}{\partial\lambda}\widetilde{P}\bigg|_{\lambda\to 1}.$$
has improved convergence by design
5. iterate until no subdivergencies left: "quasi-finite integral"

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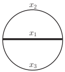
for our example only one shift is needed to arrive at quasi-finite integral



$$= \frac{\epsilon - 2}{\epsilon}\Gamma^2(1-\epsilon)\Gamma(-1+2\epsilon)\times$$
$$\times \int_0^\infty \mathrm{d}x_1 \delta(1-x_1)\int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, x_2 x_3 \left(m^2 x_1\right)^{1-2\epsilon}(x_1x_2 + x_1x_3 + x_2x_3)^{\epsilon-3}$$

Shortcomings of iterative analytic regularisation

real life problems:

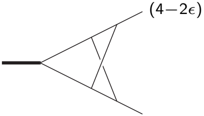
- proliferation of terms
- ambiguities
- spurious poles in ϵ

way out:

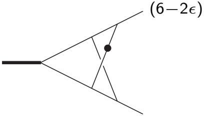
- consider full set of master integrals (basis)
- employ integration by parts (IBP) reductions

Our proposal: minimal dims & dots

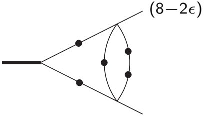
decompose wrt basis of finite integrals



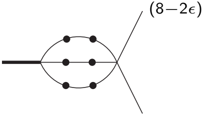
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2q^2}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2q^2}$$



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basis consists of standard Feynman integrals, but

- in shifted dimensions
- with additional dots (propagators taken to higher powers)
- old reg. shifts generated $\mathcal{O}(10\mathrm{MB})$, here: 3 lines ! (more severe at higher loops)

Existence of finite basis

1. start with some basis B for topology and subtopologies
2. assume master b not quasi-finite and has integrand

$$P = \mathcal{U}^{\nu-(L+1)\frac{d}{2}} \mathcal{F}^{-\nu+L\frac{d}{2}} \prod_{j=1}^{N} x_j^{\nu_j-1}, \qquad \text{where } \nu = \sum_{i=1}^{N} \nu_i$$

3. consider regularising dimension shift:

$$\begin{aligned} P' &= -\frac{1}{\omega_J(P)} \frac{\partial}{\partial\lambda} \widetilde{P}\Big|_{\lambda\to 1} \\ &= -\frac{1}{\omega_J(P)} \prod_{j=1}^{N} x_j^{\nu_j-1} \Bigg\{ \Big(\nu - (L+1)\tfrac{d}{2}\Big) \mathcal{U}^{(\nu+L)-(L+1)\frac{d+2}{2}} \mathcal{F}^{-(\nu+L)+L\frac{d+2}{2}} \frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\Big|_{\lambda\to 1} \\ &\qquad + \mathcal{F} \text{ derivative term} \Bigg\}, \end{aligned}$$

   with $P_{J_\lambda} = \lambda^{\deg_J(P)} \widetilde{P}$, $\mathcal{U}_{J_\lambda} = \lambda^{\deg_J(\mathcal{U})} \widetilde{\mathcal{U}}$

4. picking any monomial from $\frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\big|_{\lambda\to 1}$ or $\frac{\partial\widetilde{\mathcal{F}}}{\partial\lambda}\big|_{\lambda\to 1}$ gives

   dimension-shifted and dotted integral with improved convergence !
5. choose one term such that new integral b' is independent of $B \setminus b$
6. replace $b \to b'$ and iterate until B free of subdivergences (quasi-finite)
7. optional: transition quasi-finite $\to$ finite integrals

Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

Algorithm: construction of finite basis

- systematic scan for finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

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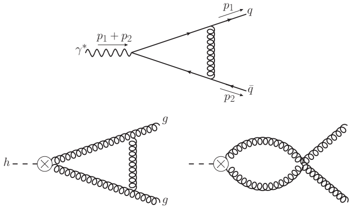
remarks:

- computationally expensive part shifted to IBP solver (`Fire`, `Reduze`, `LiteRed`)
- efficient, easy to automate (implemented in dev. version of `Reduze` 2)
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

Application: massless form factors

- massless quark and gluon form factors



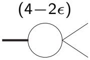
- purely virtual corrections to
  - Higgs production in gluon-fusion
  - Drell-Yan production

- simplest objects to study IR properties of QCD
  - cusp anomalous dimensions $1/\epsilon^2$: Casimir scaling ?
  - collinear anomalous dimensions $1/\epsilon$

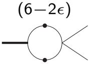
- notation: $(p_1^2 + p_2)^2 = -1$

Form factors @ 1-loop

- consider one-loop quark and gluon form factors in massless QCD
- integral basis change to finite integrals



$$= \frac{1}{\epsilon(1-\epsilon)}$$



dot: squared propagator, subscript: space-time dimension

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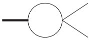

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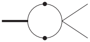

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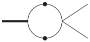
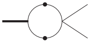
- form factors

$$\mathcal{F}_1^q(\epsilon) = C_F \frac{1}{\epsilon^2} a_1$$



$$a_1 = \frac{-2+\epsilon-2\epsilon^2}{1-\epsilon}$$

$$\mathcal{F}_1^g(\epsilon) = C_A \frac{1}{\epsilon^2} b_1$$



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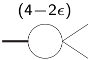
$$b_1 = \frac{-2(1-3\epsilon+2\epsilon^2+\epsilon^3)}{(1-\epsilon)^2}$$

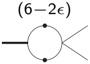
note: all divergencies explicit

Form factors @ 1-loop

- consider one-loop quark and gluon form factors in massless QCD
- integral basis change to finite integrals



$$= \frac{1}{\epsilon(1-\epsilon)}$$



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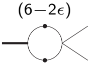
$$\mathcal{F}_1^q(\epsilon) = C_F \, \frac{1}{\epsilon^2} \, a_1$$



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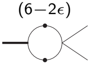

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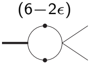
- expansion in ϵ



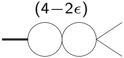
$$= 1 + \epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$a_1 = -2 - \epsilon - 3\epsilon^2 + \mathcal{O}(\epsilon^3)$$

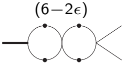
$$b_1 = -2 + 2\epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)$$

- Casimir scaling reflected by $a_1|_{\epsilon=0} = b_1|_{\epsilon=0}$

Form factors @ 2-loops: to finite basis



$$= \frac{1}{\epsilon^2}\,\frac{1}{(1-\epsilon)^2}$$



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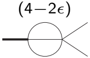
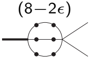


$$= \frac{1}{\epsilon}\,\frac{-4}{(2-\epsilon)^2(1-\epsilon)^2(1-2\epsilon)}$$



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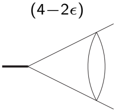
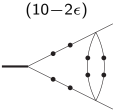


$$= \frac{1}{\epsilon^2}\,\frac{16(3-2\epsilon)(2-3\epsilon)}{(3-\epsilon)^2(2-\epsilon)^2(1-\epsilon)^3(1+2\epsilon)}$$



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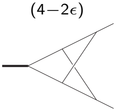
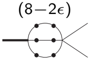
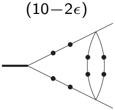


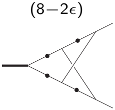
$$= \frac{1}{\epsilon^4}\,\frac{-4(2-3\epsilon)(14-81\epsilon+115\epsilon^2+14\epsilon^3-132\epsilon^4+72\epsilon^5)}{(2-\epsilon)^2(1-\epsilon)^2(1-2\epsilon)^2(2-\epsilon-2\epsilon^2)}$$



$$+ \frac{1}{\epsilon^4}\,\frac{-16(1+\epsilon)(3-2\epsilon)(2-3\epsilon)(10-61\epsilon+102\epsilon^2-44\epsilon^3-8\epsilon^4)}{(3-\epsilon)^2(2-\epsilon)^2(1-\epsilon)^3(1-2\epsilon)(1+2\epsilon)(2-\epsilon-2\epsilon^2)}$$



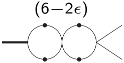
$$+ \frac{1}{\epsilon}\,\frac{4(3-4\epsilon)(1-4\epsilon)}{(2-\epsilon)(1-\epsilon)(1-2\epsilon)(2-\epsilon-2\epsilon^2)}$$



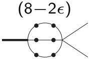
Form factors @ 2-loops

quark form factor

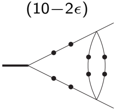
$$\mathcal{F}_2^q(\epsilon) = C_F^2 \Bigg\{ \frac{1}{\epsilon^4} \Bigg[c_1$$



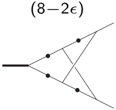
$$+ c_2$$



$$\Bigg] + \frac{1}{\epsilon^3} \Bigg[c_3$$

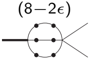


$$\Bigg] + \frac{1}{\epsilon} \Bigg[c_4$$

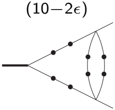


$$\Bigg] \Bigg\}$$

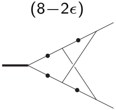
$$+ C_F C_A \Bigg\{ \frac{1}{\epsilon^4} \Bigg[c_5$$



$$+ c_6$$

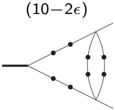


$$\Bigg] + \frac{1}{\epsilon} \Bigg[c_7$$



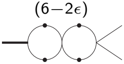
$$\Bigg] \Bigg\}$$

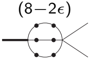
$$+ C_F N_f \Bigg\{ \frac{1}{\epsilon^3} \Bigg[c_8$$

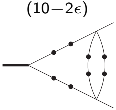


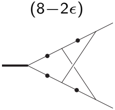
$$\Bigg] \Bigg\}$$

gluon form factor

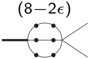
$\mathcal{F}_2^g(\epsilon) = C_A^2 \Big\{ \frac{1}{\epsilon^4} \Big[d_1$ 

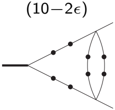
$+\, d_2$ 

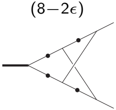
$+\, d_3$ 

$\Big] + \frac{1}{\epsilon} \Big[d_4$ 

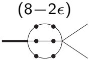
$\Big] \Big\}$

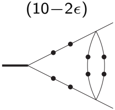
$+\, C_A N_f \Big\{ \frac{1}{\epsilon^3} \Big[d_5$ 

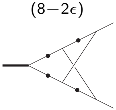
$+\, d_6$ 

$\Big] + d_7$ 

$\Big\}$

$+\, C_F N_f \Big\{ \frac{1}{\epsilon^2} \Big[d_8$ 

$+\, d_9$ 

$\Big] + d_{10}$ 

$\Big\}.$

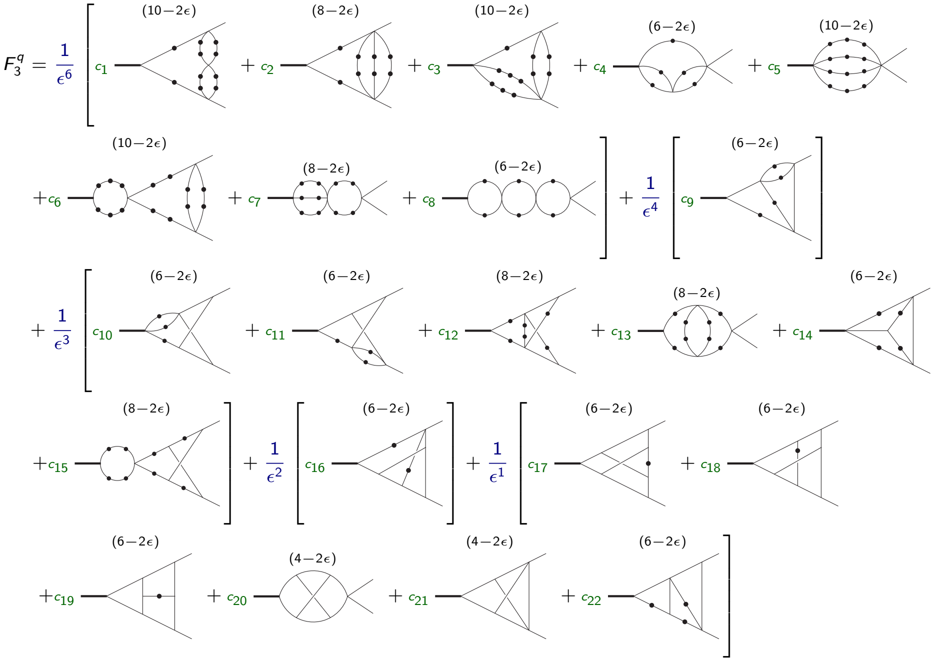
Form factors @ 3-loops

- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] $\Leftarrow$ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factors @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]

- recalculation of all results via finite integrals:
  - [AvM, Panzer, Schabinger '15]
  - automated setup, fully analytical
  - `Qgraf` [Nogueira]:
    - Feynman diagrams
  - `Reduze 2` [AvM, Studerus]:
    - interferences
    - IBP reductions
    - finite integral finder
    - basis change with dimensional recurrences
  - `HyperInt` [Panzer]:
    - integration of ϵ expanded master integrals

Quark form factor @ 3-loops [AvM, Panzer, Schabinger '15]

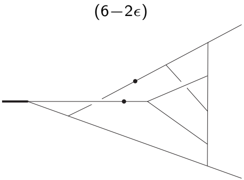


Towards the cusp anomalous dimension @ 4-loops

- cusp anomalous dimension required for N^{3}LL resummation
- Casimir scaling ?
- reduced integrand for $\mathcal{N} = 4$: [Boels, Kniehl, Yang '15]
- QCD cusp anomalous dimension:

  in our basis: no contributions from most complicated topologies through to 3-loops !

  useful also at 4-loops because not all $\mathcal{O}(300)$ master integrals linearly reducible ?
- a non-planar 12-line topology @ 4-loops [AvM, Panzer, Schabinger '15]:



$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \Big(24\zeta_2\zeta_3 + 20\zeta_5 - \frac{188}{105}\zeta_2^3 - 17\zeta_3^2 + 9\zeta_2^2\zeta_3 - 47\zeta_2\zeta_5 - 21\zeta_7 + \frac{6883}{2100}\zeta_2^4 + \frac{49}{2}\zeta_2\zeta_3^2 + \frac{1}{2}\zeta_3\zeta_5 - 9\zeta_{5,3}\Big)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

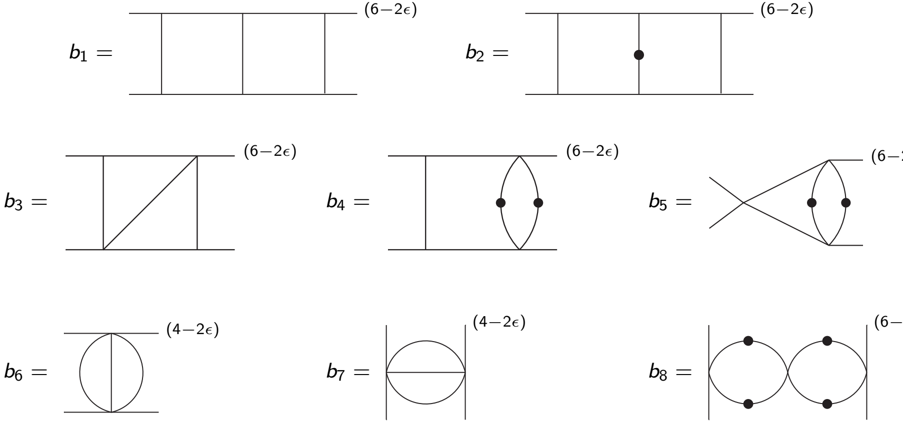
only shallow ϵ expansion needed

numerical result with `Fiesta` [A. Smirnov]: straight-forward confirmation to 4 digits

starts at weight 7, not expected to contribute to cusp anomalous dimension

Scope of the method

- "dims & dots method" general and automated
- e.g. basis of quasi-finite integrals for massless planar double boxes



- works for integrals beyond multiple polylogarithms
- works for physical kinematics

Numerical evaluations

advantages of (quasi-)finite basis:

- straight-forward to integrate numerically (in principle)
- no cancellation of spurious singularities (stability)
- no blow up in number of numerical integrations (speed, stability)
- very simple integrands also at high orders in ϵ (speed)

experiments with numerical evaluations:

- naive straight-forward implementation works already reasonably well
- convenient: employ existing sector decomposition programs
  - `Fiesta` [A. Smirnov]
  - `SecDec` [Borowka, Heinrich et al]
  - `sector_decomposition` [Bogner, Weinzierl]
- (quasi-)finite integrals: much faster & much more reliable

Conclusions

basis of finite integrals (dims and dots):

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations
- no free lunch: requires IBP reductions

results:

- massless form factors @ 3-loops: first independent rederivation at heigher weights
- result for non-planar 4-loop top level topology: starts at weight 7, irrelevant for cusp ?

outlook:

- cusp anomalous dimensions @ 4-loops
- form factors @ 4-loops
- numerical applications