Exercise 1. *Magnetic monopoles*

Consider the Lagrangian density

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{\lambda}{8}\left(\phi^2 - \eta^2\right)^2 , \tag{1}$$

where ϕ is a scalar field in the three-dimensional representation of $SO(3)$, the covariant derivative is given by

$$D_\mu\phi_a = \partial_\mu\phi_a - g\epsilon_{abc}A^b_\mu\phi_c \tag{2}$$

and the field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon_{abc}A^b_\mu A^c_\nu . \tag{3}$$

The magnetic monopole solution can be parametrised by the Ansatz

$$\phi_a = \frac{H(\xi)}{\xi}\eta\frac{x_a}{r} , \qquad A^a_i = \frac{\epsilon_{aij}x_j}{gr^2}\left(K(\xi) - 1\right) , \tag{4}$$

where $\xi = g\eta r$.

1. *Show that the covariant derivative is given in terms of this Ansatz by*

$$D_i\phi_a = \frac{K(\xi)H(\xi)}{gr^4}\left(r^2\delta_{ai} - x_a x_i\right) + \left(\xi H'(\xi) - H(\xi)\right)\frac{x_a x_i}{gr^4} . \tag{5}$$

2. *Show that the energy of the magnetic monopole is given by*

$$E = \frac{4\pi\eta}{g}\int_0^\infty \mathrm{d}\xi \frac{1}{\xi^2}\left[\frac{1}{2}(\xi H' - H)^2 + H^2K^2 + (\xi K')^2 + \frac{1}{2}(K^2-1)^2 + \frac{\lambda}{8g^2}(H^2 - \xi^2)^2\right] . \tag{6}$$

3. *Show that the energy is minimised for*

$$\begin{aligned} \xi^2 K'' &= KH^2 + K(K^2-1) , \\ \xi^2 H'' &= 2K^2H + \frac{\lambda}{2g^2}H(H^2 - \xi^2) . \end{aligned} \tag{7}$$

Solution.

1. We begin by computing the ordinary derivative of the scalar field ϕ^a_i. From the Ansatz of eq. (4) we have

$$\begin{aligned} \partial_i\phi^a =& \partial_i\frac{H}{\xi}\eta\frac{x^a}{r} = \eta\left(\frac{x_a}{\xi r}\partial_i H + \frac{Hx_a}{r}\partial_i\frac{1}{\xi} + \frac{H}{\xi}\partial_i\frac{x_a}{r}\right) \\ =& \eta\left[\left(\frac{x_a}{\xi r}H' - \frac{Hx_a}{r\xi^2}\right)\partial_i\xi + \frac{H}{\xi}\partial_i\frac{x_a}{r}\right] , \end{aligned} \tag{S.1}$$

where we have labelled $H' \equiv \partial_\xi H$. By using $\partial_i r^n = nx_i r^{n-2}$, we find

$$\partial_i\xi = g\eta\frac{x_i}{r} , \qquad \partial_i\frac{x_a}{r} = \frac{\delta_{ai}}{r} - \frac{x_i x_a}{r^3} , \tag{S.2}$$

and, thus, we obtain

$$\begin{aligned}\partial_i \phi^a =&\eta \left[\left(\frac{1}{\xi} H' - \frac{H}{\xi^2} \right) \frac{g\eta}{r^2} x_i x_a + \frac{H}{\xi} \left(\frac{\delta_{ai}}{r} - \frac{x_i x_a}{r^3} \right) \right] \\ =&\eta \left[\left(H' - 2\frac{H}{\xi} \right) \frac{x_i x_a}{r^3} + \frac{H}{\xi} \frac{\delta_{ai}}{r} \right] . \end{aligned} \tag{S.3}$$

For the coupling to the gauge field, we find

$$\begin{aligned} -g\epsilon_{abc} A_i^b \phi_c &= -g\epsilon_{abc} \frac{\epsilon_{bik} x_k}{g r^2} (K-1) \frac{H}{\xi} \eta \frac{x_c}{r} \\ &= -\eta \frac{H}{\xi} \frac{1}{r^3} (K-1)(x_i x_a - r^2 \delta_{ai}) \\ &= \eta \left[\frac{x_i x_a}{r^3} \left(-\frac{H}{\xi}(K-1) \right) + \frac{\delta_{ai}}{r} \frac{H}{\xi} (K-1) \right] , \end{aligned} \tag{S.4}$$

where, in the second equality, we have used the contraction identity $\epsilon_{abc}\epsilon_{bik} = (\delta_{ai}\delta_{ck} - \delta_{ak}\delta_{ci})$. From the sum of eq.s (S.3)-(S.4) we have

$$\begin{aligned} D_i \phi_a = \partial_i \phi_a - g\epsilon_{abc} A_i^b \phi_c = \eta \left[\frac{x_i x_a}{r^3} \left(H' - \frac{H}{\xi}(K+1) \right) + \frac{\delta_{ai}}{r} \frac{H}{\xi} K \right] \\ = \frac{1}{g r^4} \left[KH \left(r^2 \delta_{ai} - x_i x_a \right) + x_i x_a \left(\xi H' - H \right) \right] . \end{aligned} \tag{S.5}$$

2. In the case of a static field configuration, the energy is given by

$$E = -\int \mathrm{d}^3x \, \mathscr{L} = \int \mathrm{d}^3 x \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} D_i \phi^a D_i \phi^a + \frac{\lambda}{8} \left(\phi^2 - \eta^2 \right)^2 \right] . \tag{S.6}$$

Given that the fields depends exclusively on $\xi \sim r$, we can express the space integral in spherical coordinates and integrate over the angular variables. In this way, we obtain

$$E = \frac{4\pi}{g^3 \eta^2} \int \mathrm{d}\xi \, \xi^2 \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} D_i \phi^a D_i \phi^a + \frac{\lambda}{8} \left(\phi^2 - \eta^2 \right)^2 \right] . \tag{S.7}$$

We can now compute the three different terms of the Lagrangian individually.

The potential term is simply given by

$$-\frac{\lambda}{8} \left(\phi^2 - \eta^2 \right)^2 = -\frac{\lambda}{8} \frac{\eta^4}{\xi^4} \left(H^2 - \xi^2 \right)^2 . \tag{S.8}$$

For the kinetic term we observe that

$$\begin{aligned} (x_a x_i)^2 =& r^4 , \\ (r^2 \delta_{ai} - x_i x_a)^2 =& 3r^4 - 2r^2 + r^2 = 2r^2 , \\ (r^2 \delta_{ai} - x_i x_a) x_a x_i =& r^4 - r^4 = 0 . \end{aligned} \tag{S.9}$$

Therefore, from the result of eq. (S.5), we have

$$\frac{1}{2} D_i \phi^a D_i \phi^a = \frac{1}{2} \frac{g^2 \eta^4}{\xi^4} \left(2K^2 H^2 + (\xi H' - H)^2 \right) . \tag{S.10}$$

For the pure gauge part of the Lagrangian, we start by constructing the field strength tensor $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a - g\epsilon_{abc} A_i^a A_j^c$.

The derivative of the gauge field is given by

$$\begin{aligned}\partial_i A^a_j &= \partial_i \frac{\epsilon_{ajk} x_k}{g r^2}(K-1) = \frac{\epsilon_{ajk}}{g}\left[(K-1)\partial_i \frac{x_k}{r^2} + \frac{x_k}{r^2}\partial_i K\right] \\ &= \frac{\epsilon_{ajk}}{g}\left[(K-1)\partial_i \frac{x_k}{r^2} + \frac{x_k}{r^2}K'\partial_i \xi\right] \\ &= \frac{\epsilon_{ajk}}{g}\left[(K-1)\left(\frac{\delta_{ik}}{r^2} - 2\frac{x_i x_k}{r^4}\right) + g\eta\frac{x_i x_k}{r^3}K'\right] \\ &= -\frac{1}{gr^2}\epsilon_{aij}(K-1) + \frac{1}{gr^4}x_i\epsilon_{ajk}x_k\left(\xi K' - 2(K-1)\right) . \end{aligned} \tag{S.11}$$

In a similar way, we have

$$\partial_j A^a_i = \frac{1}{gr^2}\epsilon_{aij}(K-1) + \frac{1}{gr^4}x_j\epsilon_{aik}x_k\left(\xi K' - 2(K-1)\right) . \tag{S.12}$$

Therefore, the Abelian part of the field strength is

$$\partial_i A^a_j - \partial_j A^a_i = -\frac{2}{gr^2}\epsilon_{aij}(K-1) + \frac{1}{gr^4}(x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k)(\xi K' - 2(K-1)) . \tag{S.13}$$

For the non-Abelian part, we have

$$\begin{aligned} -\epsilon_{abc}A^b_i A^c_j &= -\frac{1}{g^2 r^4}(K-1)^2\epsilon_{abc}\epsilon_{bik}x_k\epsilon_{cjm}x_m \\ &= -\frac{1}{g^2 r^4}(K-1)^2(\delta_{ci}\delta_{ak} - \delta_{ck}\delta_{ai})x_k\epsilon_{cjm}x_m \\ &= -\frac{1}{g^2 r^4}(K-1)^2 x_a\epsilon_{ijm}x_m . \end{aligned} \tag{S.14}$$

By adding up eq.s (S.13)-(S.14), we arrive at

$$\begin{aligned} F^a_{ij} = &-\frac{2}{gr^2}(K-1)\epsilon_{aij} + \frac{1}{gr^4}\left(\xi K' - 2(K-1)\right)(x_i\epsilon_{ajk}x_k - x_j\epsilon_{aim}x_m) \\ &- \frac{1}{gr^4}(K-1)^2 x_a\epsilon_{ijm}x_m . \end{aligned} \tag{S.15}$$

In the computation of the square of the field strength, the following contractions appear:

$$\begin{aligned} \epsilon_{aij}\epsilon_{aij} &= 6 & x_i\epsilon_{ajk}x_k x_i\epsilon_{ajm}x_m &= 2r^4 \\ x_i\epsilon_{ajk}x_k x_j\epsilon_{ajm}x_m &= 0 & (x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k)^2 &= 4r^4 \\ x_a x_a\epsilon_{ijm}\epsilon_{ijk}x_m x_k &= 2r^4 & \epsilon_{aij}(x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k) &= -4r^2 \\ \epsilon_{aij}x_a\epsilon_{ijm}x_m &= 2r^2 & (x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k)x_a\epsilon_{ijm}x_m &= 0 . \end{aligned} \tag{S.16}$$

By making use of these identities we arrive at

$$\begin{aligned} \frac{1}{4}F^a_{ij}F^a_{ij} =& \frac{6}{g^2 r^4}(K-1)^2 + \frac{1}{2g^2 r^4}(K-1)^4 + \frac{1}{g^2 r^4}(\xi K' - 2(K-1))^2 \\ &+ \frac{4}{g^2 r^4}(K-1)(\xi K' - 2(K-1)^2) + \frac{2}{g^2 r^4}(K-1)^3 \\ =& \frac{1}{2g^2 r^4}\left[4(K-1)^2 + (K-1)^4 + 4(K-1)^3 + 2(\xi K')^2\right] \\ =& \frac{1}{2g^2 r^4}\left[(K^2-1)^2 + 2(\xi K')^2\right] \\ =& \frac{g^2\eta^4}{2\xi^4}\left[(K^2-1)^2 + 2(\xi K')^2\right] , \end{aligned} \tag{S.17}$$

where, in the last step, we have used

$$\begin{aligned}4(K-1)^2+(K-1)^4+4(K-1)^3 =&(K-1)^2((K-1)^2+4K)\\ =&(K-1)^2(K+1)^2=(K^2-1)^2\,.\end{aligned} \tag{S.18}$$

If we now add we add eq.s (S.8)-(S.10)-(S.14), the energy becomes

$$E= \ =\frac{4\pi\eta}{g}\int\frac{\mathrm{d}\xi}{\xi^2}\left[\frac{1}{2}(K^2-1)^2+(\xi K')^2+K^2H^2+\frac{1}{2}(\xi H'-H)^2+\frac{\lambda}{8g^2}(H^2-\xi^2)^2\right]. \tag{S.19}$$

3. The energy of the system is a functional of the shape functions $K(\xi)$ and $H(\xi)$,

$$E[g]=\frac{4\pi\eta}{g}\int\mathrm{d}\xi\,\mathscr{E}(g(\xi),g'(\xi))\,,\qquad g=H,K\,, \tag{S.20}$$

where, according to eq. (S.19), we have

$$\mathscr{E}(g(\xi),g'(\xi))=\frac{1}{\xi^2}\left[\frac{1}{2}(K^2-1)^2+(\xi K')^2+K^2H^2+\frac{1}{2}(\xi H'-H)^2+\frac{\lambda}{8g^2}(H^2-\xi^2)^2\right]. \tag{S.21}$$

We determine the minimum energy configuration by demanding

$$\frac{\delta E}{\delta K}=\frac{\delta E}{\delta H}=0\,. \tag{S.22}$$

Given an infinitesimal deformation $g(\xi)\to g(\xi)+\epsilon\phi(\xi)$, the corresponding energy variation can be written as

$$\begin{aligned}\frac{\delta E}{\delta g}=&\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int d\xi\left[\mathscr{E}(g(\xi)+\epsilon\varphi(\xi),g'(\xi)+\epsilon\varphi'(\xi))\right]\Big|_{\epsilon=0}\\ =&\int d\xi\left[\frac{\partial\mathscr{E}(g,g')}{\partial g}\varphi+\frac{\partial\mathscr{E}(g,g')}{\partial g'}\varphi'\right]\\ =&\int d\xi\left[\frac{\partial\mathscr{E}(g,g')}{\partial g}-\frac{\partial}{\partial\xi}\frac{\partial\mathscr{E}(g,g')}{\partial g'}\right]\varphi\,,\end{aligned} \tag{S.23}$$

where, in the last step, we have integrated by parts the second term under the assumption that boundary term vanishes. Hence, for an arbitrary field variation $\phi(\xi)$ we must require

$$\frac{\partial\mathscr{E}(g,g')}{\partial g}-\frac{\partial}{\partial\xi}\frac{\partial\mathscr{E}(g,g')}{\partial g'}=0\,,\qquad g=H,K\,. \tag{S.24}$$

The derivatives of $\mathscr{E}$ are easily checked to be

$$\begin{aligned}\frac{\partial\mathscr{E}}{\partial K}=&\frac{2}{\xi^2}\left(K(K^2-1)+KH^2\right)\,,\\ \frac{\partial\mathscr{E}}{\partial H}=&\frac{1}{\xi^2}\left(2K^2H-\xi H'+H+\frac{\lambda}{2g^2}H(H^2-\xi^2)\right)\,,\\ \frac{\partial\mathscr{E}}{\partial K'}=&2K'\qquad\Rightarrow\qquad\frac{\partial}{\partial\xi}\frac{\partial\mathscr{E}}{\partial K'}=2K''\\ \frac{\partial\mathscr{E}}{\partial H'}=&H'-\frac{H}{\xi}\ \ \Rightarrow\qquad\frac{\partial}{\partial\xi}\frac{\partial\mathscr{E}}{\partial H'}=H''-\frac{H'}{\xi}+\frac{H}{\xi^2}\,,\end{aligned} \tag{S.25}$$

so that minimum energy conditions of eq. (S.24) correspond to second order differential equations for the functions $K(\xi)$ and $H(\xi)$,

$$\begin{aligned}\frac{\delta E}{\delta K} = 0 \qquad &\Rightarrow \qquad \xi^2 K'' = \left(K(K^2-1) + KH^2\right) \, , \\ \frac{\delta E}{\delta H} = 0 \qquad &\Rightarrow \qquad \xi^2 H'' = 2K^2 H + \frac{\lambda}{2g^2} H(H^2 - \xi^2) \, . \end{aligned} \tag{S.26}$$

The (difficult) analytic solution of these differential equations is greatly simplified in the limit $\lambda/g^2 \sim 0$. Under this assumption, we can verify that eq. (S.26) is solved by

$$H(\xi) = \xi \coth \xi - 1 \, , \qquad K(\xi) = \xi \operatorname{cosech} \xi \, , \, , \tag{S.27}$$

and that the monopole energy integrates to $E = 4\pi\eta/g$. In this limit, $K(\xi) \to 0$ for $\xi \to \infty$ and the long distance behaviour of the gauge field defined in eq. (4) is

$$A_i^a \sim -\frac{\epsilon_{aij} x_j}{g r^2} \, , \tag{S.28}$$

so that the magnetic field becomes

$$B_i^a = (\nabla \times A^a)_i = \frac{x_i x_a}{g r^4} \, . \tag{S.29}$$

After the spontaneous symmetry breaking, $SO(3)$ is reduced to the $U(1)$ group of rotation about the radial direction. Hence, the magnetic field can be identified wth the component of B_i^a along this direction,

$$\mathbf{B} = \frac{\hat{\mathbf{r}}}{g r^2} \, , \tag{S.30}$$

that corresponds to the magnetic field of a monopole with magnetic charge $4\pi/g$.