MMP II
Exercise Sheet 12

FS 19
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Exercise 1 [The orthogonal and unitary groups (5 points)]
Proof that O(n) is a subgroup of U(n).

Which of the following statements is correct?

- O(n) $\triangleleft$ U(n)
- SO(n) $\triangleleft$ O(n) $\leq$ U(n)
- SO(n) $\triangleleft$ O(n) $\triangleleft$ U(n)
- SO(n) $\leq$ O(n) $\leq$ SU(n) $\leq$ U(n)

Is any of these groups abelian?

Exercise 2 [Lie groups and their Lie algebras (5 points)]
Let G be a matrix Lie group and $[X, Y] := XY - YX$ the commutator.
Show that $(T_eG, ad) = (Lie(G), [\cdot, \cdot])$, where

$$T_eG := \left\{ \frac{d}{dt}A(t)\bigg|_{t=0} \middle| A : (-\epsilon, \epsilon) \to G, \epsilon > 0, A(0) = e \right\}, \quad \begin{array}{rccc} \mathrm{Ad}: & G & \to & \mathrm{Aut}(T_eG) \\ & A & \mapsto & (X \mapsto AXA^{-1}) \end{array}$$

$$\text{and } ad(X)(Y) := \frac{d}{dt}\mathrm{Ad}(A(t))(Y)|_{t=0}, \quad A(0) = e, \frac{d}{dt}A(t)|_{t=0} = X, \quad X, Y \in T_eG$$

Write down a physical basis of the Lie algebra $\mathfrak{so}^+(1,3)$ of the Lorentz group
$\mathrm{SO}^+(1,3) := \{\Lambda \in \mathbb{R}^{4\times 4} | \Lambda^T \eta \Lambda = \eta, \det \Lambda = 1\}$, where η is the Minkowski-metric.

Hint: Boosts and rotations.

Exercise 3 [Transformation rules of tensors (5 points)]
From linear algebra you know, that every scalar product can be written as a matrix product: $\langle x|y\rangle_M = x^T M y$, where M is symmetric. This is a bi-linear map $\langle\cdot|\cdot\rangle_M : V \times V \mapsto \mathbb{R}$.
If we write the inner product using Einstein summation convention, we get:

$$\langle x|y\rangle_M = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i)^T M_{ij} y_j = x_i M^i_j y^j.$$

What sort of tensors are $x^T = (x_i)_{i=1}^n$, $y = (y^i)_{i=1}^n$ and $M = (M^i_j)_{i,j=1}^n$?

To map two vectors to a scalar we have to change the tensor type of x via the transpose operation: $x^T = (x^i)^T = \delta_{ij}x^j = x_i$
We can circumvent this by using a different type of tensor instead of the matrix.

Consider $n = 4$ and a scalar product on vector fields $X(x)$ and $Y(x)$ given by $\langle X|Y\rangle_{g(x)} = g_x(X,Y) = g_{\mu\nu}(x)X^\mu Y^\nu$ with signature $(1,3) = (-,+,+,+)$, which is dependent on the position $x = (x^\mu)_{\mu=0}^3$.

Prove, that the metric $g_{\mu\nu}(x)$ is a tensor field. What sort of tensor field is it?

Hint: Transform the two vector fields X^μ, Y^μ in the scalar product to get a transformation formula for the coefficients $g_{\mu\nu}$.

The Christoffel symbol is defined as: $\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}(g_{\nu\rho,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu})$, where the coma is a short hand for partial derivative $g_{\rho\sigma,\nu} = \frac{\partial}{\partial x^\nu}g_{\rho\sigma}(x^\mu)$.

Check, whether the following operations on tensor fields, which we used to define $\Gamma^\mu_{\rho\sigma}$, map to tensor fields.

- $g_{\mu\nu}(x) + h_{\mu\nu}(x)$
- $\frac{\partial}{\partial x^\mu}g_{\rho\sigma}(x)$
- $(g^{-1}g)(x) = g^{\mu\eta}(x)g_{\eta\nu}(x)$

Hint: Use the transformation properties of tensors given in the script (5.1.4), p. 100. The partial derivative transforms as a contra-variant vector:

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (\partial'_\mu x^\nu)\partial_\nu$$

Do the Christoffel symbols form a tensor field?