Exercise 1. *Parity conservation in QCD*

Consider the QCD Lagrangian with massive fermions,

$$\mathscr{L}^{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} - \bar{q}_i \left(\not{D} + m_i \right) q_i + \theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu,a} \,, \tag{1}$$

where all the masses m_i are real (i.e. we have chosen $\theta = \bar{\theta}$).

In a volume V, the Euclidean path-integral formula for the ground-state energy is then

$$e^{-VE} = \int \mathscr{D}A \mathscr{D}\bar{q} \mathscr{D}q \exp\left(- \int \mathrm{d}^4 x_E \; \mathscr{L}_E^{QCD} \right) , \tag{2}$$

where the subscript E indicates Euclidean quantities.

1. *By integrating out the fermions, show that the energy is minimised for $\theta = 0$.*
2. *Show that this argument holds for any parity-breaking operator, not only the θ-term (and in particular for the axion).*

Proceed as follows:

- *First, show that the integrand in the Euclidean path-integral is given by*

$$\exp\left[- \int \mathrm{d}^4 x_E \left(\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \bar{q}_i (\not{D} + m_i) q_i + \theta \frac{g^2}{32\pi^2} i \epsilon^{\mu\mu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \right) \right] .$$

- *Next, integrate out the fermions so to obtain the fermion determinant $\det(\not{D} + m)$. Use $\gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5$ to show that the non-zero eigenvalues of $i\not{D}$ are paired and use this observation to prove that the fermion determinant is positive.*
- *Finally, determine the minimum of the energy as a function of θ.*

Solution. The derivation follows [VW84a], which closely references [VW84b] for the positivity of the fermion determinant. The conventions for QCD in Minkowski space are the same as [Wei95].

In the following, we parametrise the QCD Lagrangian as

$$\mathscr{L}_\lambda = \mathscr{L}^{(\lambda=0)}_{\text{QCD}} + \lambda X \,, \quad \text{with} \quad \lambda = \theta \frac{g^2}{32\pi^2} \in \mathbb{R} \,, \tag{S.1}$$

where X is a hermitian, parity-odd operator that we will take to be $X = \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$. We show that the presence of a parity-odd term in the Lagrangian, $\lambda \neq 0$, increases the ground-state energy of the system,

$$E_0(\lambda) > E_0(0) \,. \tag{S.2}$$

The most important consequence of eq. (S.2) is that parity cannot be spontaneously broken. In fact, if we work at first order in λ, we can write $E_0(\lambda)$ as

$$E_0(\lambda) = E_0(0) + \lambda \int \mathrm{d}^3 x \, \langle X \rangle + \mathscr{O}\left(\lambda^2\right) \,, \tag{S.3}$$

where $\langle X \rangle$ denotes a vacuum expectation value of X in the theory with $\lambda = 0$. Parity is spontaneously broken if $\langle X \rangle \neq 0$. However, we observe that, since X is an odd operator, nonzero values of $\langle X \rangle$ must come in pairs: if $\langle X \rangle \neq 0$, then there are at least two vacua related by parity, one with expectation value $\langle X \rangle$ and one with expectation value $-\langle X \rangle$. Therefore, regardless of the sign of λ, if $\langle X \rangle \neq 0$ there is certainly a vacuum state with $E_0(\lambda) < E_0(0)$, which contradicts eq. (S.2) .

In order to prove that $\lambda = 0$ minimises the ground-state energy, we start from the the path-integral formula for $E_0(\lambda)$ in the four-dimensional Euclidean volume V_4,

$$e^{-V_4 E_0(\lambda)} = \int \mathscr{D}\phi e^{S_E[\phi]}\,, \tag{S.4}$$

where ϕ denotes all the fields of the theory and $S_E[\phi] = \int \mathrm{d}^4 x_E \mathscr{L}_E(\phi)$ is the Euclidean action. Eq. (S.4) can be obtained from the Minkowskian action,

$$e^{-iTV_3 E_0} = \int \mathscr{D}\phi e^{iS[\phi]}\,, \tag{S.5}$$

by analytic continuation. Since we are working in the metric $\eta^{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$, this amounts to performing a Wick rotation of the time coordinate $x^0 = -ix^4$, which implies $TV_3 = \Delta x^0 V_3 = -iV_4$. The rules for the the analytic continuation onto the negative imaginary x^0-axis are fixed as :

$$\begin{aligned} x^0 &= -ix^4 = -ix_4\,, \quad x_0 = ix_4 = ix^4\,, \\ A^0 &= -iA^4 = -iA_4\,, \quad A_0 = iA^4 = iA_4\,, \\ \partial^0 &= -i\partial^4 = -i\partial_4\,, \quad \partial_0 = i\partial^4 = i\partial_4\,, \end{aligned} \tag{S.6}$$

The gamma matrices satisfy $\gamma_0^\dagger = -\gamma_0$, $\gamma_k^\dagger = \gamma_k$ and, for the analytic continuation, we fix $\gamma^0 = i\gamma^4 = i\gamma_4$. We denote $\bar\psi = \psi^\dagger \gamma^0 = \psi^\dagger(i\gamma^4)$ throughout.

With the above conventions it is easy to verify that, under Wick rotation, $F_{0k} \to iF^E_{4k}$ and $F^{0k} \to -iF^E_{4k}$. Thus, we have

$$F^a_{\mu\nu} F^{\mu\nu,a} \to \mathrm{Tr}\, F^{E,a}_{\mu\nu} F^{E,a}_{\mu\nu}\,, \tag{S.7}$$

whereas, for the parity-odd operator we obtain

$$F^a_{\mu\nu} \tilde F^{\mu\nu,a} \to \pm i F^{E,a}_{\mu\nu} \tilde F^{E,a}_{\mu\nu}\,. \tag{S.8}$$

Not that the (irrelevant) sign on the r.h.s. of eq. (S.8) depends on the convention adopted for ϵ_{1234}. Finally, because of $\gamma^0 D_0 = (i\gamma^4)(iD_4)$, for the Dirac operator we have

$$\bar\psi(\gamma^\mu D_\mu + M)\psi \to \bar\psi(\gamma_\mu D_\mu + M)\psi\,, \tag{S.9}$$

where M is the definte positve mass matrix. Hence, the Minkowskian action

$$iS[\phi] = i\int \mathrm{d}^4 x \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} - \bar\psi(\not{D} + M)\psi + \lambda F^a_{\mu\nu} \tilde F^{\mu\nu,a} \right] \tag{S.10}$$

becomes

$$iS[\phi] \to -\int \mathrm{d}^4 x \left[\frac{1}{4} F^{E,a}_{\mu\nu} F^{E,a}_{\mu\nu} + \bar\psi(\not{D} + M)\psi \mp i\lambda F^{E,a}_{\mu\nu} \tilde F^{E,a}_{\mu\nu} \right] = S_E[\phi] \tag{S.11}$$

The ground-state energy is then

$$e^{-V_4 E_0(\lambda)} = \int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} \exp\left[-\int \mathrm{d}^4x \left(\frac{1}{4}F^{E,a}_{\mu\nu}F^{E,a}_{\mu\nu} + \bar{\psi}(\not{D} + M)\psi \mp i\lambda F^{E,a}_{\mu\nu}\tilde{F}^{E,a}_{\mu\nu}\right)\right] . \tag{S.12}$$

The integration over the spinor field configurations leads to the fermion determinant

$$\int \mathscr{D}\psi \mathscr{D}\bar{\psi} \exp\left(-\int \mathrm{d}^4x_E \bar{\psi}(\not{D} + M)\psi\right) = \det(\not{D} + M)$$

and it gives us

$$e^{-V_4 E_0(\lambda)} = \int \mathscr{D}A_\mu \det\left(\not{D} + M\right) \exp\left\{-\int \mathrm{d}^4x_E \left(\frac{1}{4}F^{E,a}_{\mu\nu}F^{E,a}_{\mu\nu} \mp i\lambda F^{E,a}_{\mu\nu}\tilde{F}^{E,a}_{\mu\nu}\right)\right\} .$$

At this level, if we assume the positivity of the Euclidean integration measure, we can determine the effect of the real parameter λ on the ground-state energy by studying the positivity properties of the integrand. To this aim, we observe that, since QCD is a theory with vector-like coupling between fermions and gauge bosons, it does not distinguish between left-handed and right-handed quarks. This implies that the eigenvalues of $i\not{D}$ come in pairs of opposite signs. In fact, if we suppose ψ to be an eigenstate of $i\not{D}$,

$$i\not{D}\psi = \alpha\psi \,, \quad \text{with} \quad \alpha \in \mathbb{R}, \tag{S.13}$$

than, from the anticommutation of γ^5, it follows immediately that $\gamma^5\psi$ is another eigenstate, with opposite eigenvalue,

$$i\not{D}(\gamma^5\psi) = -\gamma_5(i\not{D})\psi = -\alpha(\gamma_5\psi) \,. \tag{S.14}$$

Therefore, by diagonalising the operator $(\not{D} + M)$, we can write the fermion determinant as

$$\det(\not{D} + M) = \prod_{\alpha\in\mathbb{R}}(M - i\alpha) = \prod_{\alpha>0}(M + i\alpha)(M - i\alpha) = \prod_{\alpha>0}(M^2 + \alpha^2) \,, \tag{S.15}$$

which shows that the determinant is a positive function of the gauge field A_μ. Note that in eq. (S.15) we have omitted possible zero eingevalues of $\not{D}$. Nevertheless, they would contribute with overall factors M, which we have assumed to be positive definite and, hence, they would not modify the sign of the determinant.

The positivity of the fermion determinant, together with $F^{E,a}_{\mu\nu}F^{E,a}_{\mu\nu} \geq 0$, proves that the integrand of the theory with $\lambda = 0$ is positive. In addition, since we also have $F^E_{\mu\nu}\tilde{F}^E_{\mu\nu} \geq 0$, the presence of $\lambda \neq 0$ adds a phase to the exponential which, in turn, can only make the integral over the field configurations smaller. Hence, we have found

$$e^{-V_4 E_0(0)} > e^{-V_4 E_0(\lambda)} \,, \tag{S.16}$$

which implies, as expected,

$$E_0(\lambda) > E_0(0) \,. \tag{S.17}$$

From the above consideration we conclude that $X = \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ cannot get a vacuum expectation value. Therefore if, as in the axion theory, the coefficient λ is effectively dynamical, QCD will choose a vacuum state with $\lambda = 0$. Note that same reasoning applies to all parity-odd terms built from the gauge field, since any parity-odd operator must involve an odd power of terms with exactly one F_{0i}, which gives rise to a phase when continued to Euclidean space.

Exercise 2. ***Electric dipole moment of the neutron***

The electric dipole moment of the neutron, denoted by d_E*, is defined as*

$$\langle n(p')|J_\mu^{em}|n(p)\rangle\big|_{edm} = id_E\ \bar{u}(p')\sigma_{\mu\nu}(p'-p)^\nu\gamma^5 u(p)\,. \tag{3}$$

No electric diplole moment is produced at first order in the Standard Model. The main contribution to d_E *comes from the* θ *parameter that appears in the full Lagrangian of QCD:*

$$\mathscr{L}_{QCD} = \mathscr{L}_{QCD}^{(\theta=0)} + \theta\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu,a}\,. \tag{4}$$

Estimate the neutron electric dipole moment d_E *as a function of* $\bar{\theta} := \theta - \arg\det M$*, where* M *denotes the quark mass matrix.*

Solution. The two classic papers calculating the relation between $\bar{\theta}$ and the neutron electric dipole moment are [Bal79] and [CVVW79]. Their results are pretty similar and we follow the former[1].

We start from the three-flavour QCD Lagrangian with real mass term ($\bar{\theta} := \theta - \arg\det M = \theta$),

$$\mathscr{L}_M^{QCD} = \sum_{k=u,d,s} m_k\bar{\psi}_k\psi_k = \bar{\psi}M\psi\,, \quad M = \mathrm{diag}\,(m_u, m_d, m_s)\,, \tag{S.18}$$

and we transform away the anomaly term $\mathscr{L}_\theta^{QCD}$, by means of a chiral rotation of the quark fields,

$$\begin{aligned}\psi(x) =& \psi_R + \psi_L \to e^{i\alpha\gamma_5}\psi(x) = (e^{i\alpha}\psi_R + e^{-i\alpha}\psi_L)\,,\\ \bar{\psi}(x) =& \bar{\psi}_R + \bar{\psi}_L \to \psi(x)e^{-i\alpha\gamma_5} = (e^{-i\alpha}\bar{\psi}_R + e^{i\alpha}\bar{\psi}_L)\,.\end{aligned} \tag{S.19}$$

$\mathscr{L}_M^{QCD}$ is not invariant under chiral rotations and it acquires an anomalous (imaginary) term,

$$\mathscr{L}_M^{QCD} \to \bar{\psi}_L e^{2i\alpha}M\psi_R + \text{h.c.} \tag{S.20}$$

Under the assumption $\alpha \ll 1$, we parametrise the new mass matrix as $\widetilde{M} = e^{2i\alpha}M = M + i\eta$ and we obtain

$$\begin{aligned}\mathscr{L}_{\widetilde{M}}^{QCD} =& \bar{\psi}_L(M+i\eta)\psi_R + \bar{\psi}_R(M-i\eta)\psi_L\\ =& \bar{\psi}_L M\psi_R + \bar{\psi}_R M\psi_L + i\eta(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L)\\ =& \mathscr{L}_M^{QCD} + \mathscr{L}_{CP}\,,\end{aligned} \tag{S.21}$$

where we have defined

$$\mathscr{L}_{CP} = i\eta(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L) = i\eta\bar{\psi}\gamma^5\psi\,. \tag{S.22}$$

The parameter η of the CP-violating term of the Lagrangian is related to the quark masses through the relation $\bar{\theta} = -\arg\det\tilde{M}$,

$$\begin{aligned}\bar{\theta} &= -\arg\det(M+i\eta) = -\arg\left[(m_u+i\eta)(m_d+i\eta)(m_s+i\eta)\right]\\ &\approx -\arg\left[m_u m_d m_s + i\eta(m_u m_d + m_d m_s + m_s m_u)\right]\\ &= -\arctan\left[\eta\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)\right] \approx -\eta\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)\,.\end{aligned} \tag{S.23}$$

[1]Note that the conventions used for $\bar{\theta}$ in [Bal79] are not the same as the ones used in the lecture. In our notation, the reference uses

$$\bar{\theta} = \frac{1}{n_f}\arg\det\tilde{M}$$

with n_f the number of light quark flavours which leads to the above difference in $\mathscr{L}_{CP}$.

Therefore, we have

$$\mathscr{L}_{CP}^{QCD} = -i\bar{\theta}\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)^{-1}\bar{\psi}\gamma^5\psi\,. \tag{S.24}$$

$\mathscr{L}_{CP}^{QCD}$ can induce a non-zero contribution to the neutron electric dipole moment,

$$d_E = \langle n|\int \mathrm{d}^3x \mathscr{L}_{CP}^{QCD}|n\rangle\,. \tag{S.25}$$

If interpret $\mathscr{L}_{CP}^{QCD}$ as a small perturbation to the CP-conserving Lagrangian, we can work in time-independent perturbation theory up to first order in $\bar{\theta}$ and write the perturbed neutron state as

$$|n\rangle = |N_0\rangle + \sum_{m>0}\frac{\langle N_m|\int \mathrm{d}^3x\mathscr{L}_{CP}^{QCD}|N_0\rangle}{E_m-E_0}|N_m\rangle\,, \tag{S.26}$$

where N_0 is the neutron ground-state of the CP-conserving theory and N_m are the parity-odd excited states. With eq. (S.26), d_E becomes

$$\begin{aligned} d_E = \langle n\,|d|\,n\rangle &= 2\sum_{m>0}\frac{1}{E_m-E_0}\Re\left(\langle N_0|d|N_m\rangle\langle N_m|\int \mathrm{d}^3x\mathscr{L}_{CP}^{QCD}|N_0\rangle\right) \\ &= -2\bar{\theta}\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)^{-1}\sum_{m>0}\frac{1}{E_m-E_N}\Re\left(\langle N_0|d|N_m\rangle\langle N_m|\int \mathrm{d}^3x\, i\bar{\psi}\gamma_5\psi|N_0\rangle\right). \end{aligned} \tag{S.27}$$

We estimate the order of magnitude of d_E by restricting the sum to the lowest-lying resonance N_1, with mass $m_{N_1} = 1535$ MeV,

$$\frac{1}{E_1-E_0} \approx \frac{1}{1535\,\text{MeV} - 940\,\text{MeV}}\,. \tag{S.28}$$

In addition, we have (from the numerical values of [ea08])

$$m_s \approx 105\text{MeV}\,,\quad \frac{m_s}{m_d}\approx 20\,,\quad \frac{m_d}{m_u}\approx 2\,,\quad \langle N_0|d|N_m\rangle \approx e\cdot r_P \approx e\cdot 0.9\cdot 10^{-13}\text{cm}\,, \tag{S.29}$$

where r_P denotes the proton charge radius. Finally, if we consider $\langle N_m|\int \mathrm{d}^3x\, i\bar{\psi}\gamma_5\psi|N_0\rangle \approx 1$ (the actual calculation of the matrix elements can be done using the MIT bag model, cfr [Bal79]), we obtain

$$d_E \approx 5\cdot 10^{-16}\left|\bar{\theta}\right| e\,\text{cm}\,. \tag{S.30}$$

From the experimental bound $d_E^{\text{exp}} < 0.3\cdot 10^{-23} e\,\text{cm}$ we have, therefore, $\left|\bar{\theta}\right| \lessapprox 0.6\cdot 10^{-8}$.

References

[Bal79] *Varouzhan Baluni. cp-nonconserving effects in quantum chromodynamics.* Phys. Rev. D*, 19(7):2227–2230, Apr 1979.*

[CVVW79] *R.J. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten. Chiral estimate of the electric dipole moment of the neutron in quantum chromodynamics.* Physics Letters B*, 88(1-2):123 – 127, 1979.*

[ea08] *C. Amsler et al. Review of particle physics.* Physics Letters B*, 667(1-5):1 – 6, 2008. Review of Particle Physics.*

[VW84a] *Cumrun Vafa and Edward Witten. Parity conservation in quantum chromodynamics.* Phys. Rev. Lett.*, 53(6):535–536, Aug 1984.*

[VW84b] *Cumrun Vafa and Edward Witten. Restrictions on symmetry breaking in vector-like gauge theories.* Nuclear Physics B*, 234(1):173 – 188, 1984.*

[Wei95] *Steven Weinberg.* The quantum theory of fields*. Cambridge University Press, 1995.*