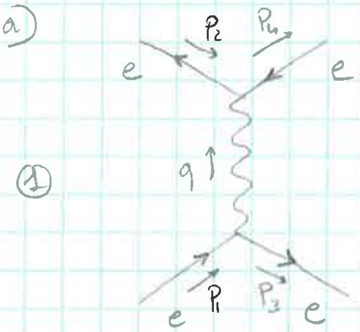
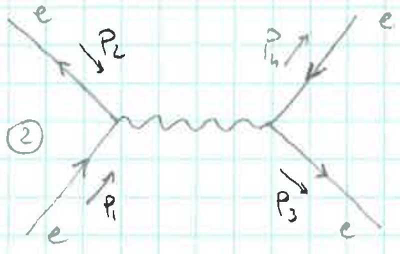
ELECTRON POSITRON SCATTERING

a)





b) there are two diagrams, hence 2 contributions to the amplitude

$$M_1 = (2\pi)^4 \cancel{(2\pi)^4} \int \frac{d^4q}{\cancel{(2\pi)^4}} \delta^4(p_1 - p_3 - q)\,\delta^4(p_2 + q - p_4)$$
$$\times \left[\bar{v}(2)\,(i g_e \gamma^\mu)\,v(4)\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{u}(3)\, i g_e \gamma^\nu u(1)\right] d^4q$$

adjoint / matrix / spinor

$\downarrow$ photon propagator

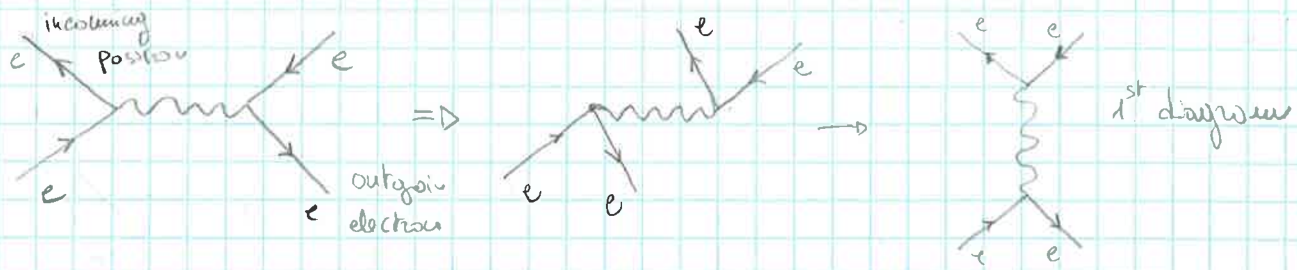
N.B. Proceeding backwards along the antiparticle means to proceed forward in time.

$$M_1 = -\frac{g_e^2}{(p_1 - p_3)^2} \left[\bar{v}(2)\,\gamma^\mu v(4)\right]\left[\bar{u}(3)\,\gamma_\mu u(1)\right]$$

$$M_2 = (2\pi)^4 \int d^4q\; \delta^4(q - p_3 - p_4)\,\delta^4(p_1 + p_2 - q) \times$$
$$\times \left[\bar{u}(3)\, i g_e \gamma^\mu v(4)\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{v}(2)\, i g_e \gamma^\nu u(1)\right]$$

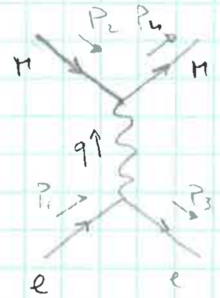
$$M_2 = -\frac{g_e^2}{(p_1 + p_2)^2} \left[\bar{u}(3)\,\gamma^\mu v(4)\right]\left[\bar{v}(2)\,\gamma_\mu u(1)\right]$$

For the antisymmetrization rule we need to subtract the two amplitudes.



$$M = M_1 - M_2$$

c) ELECTRON - MUON SCATTERING



Walking backward along each fermion line

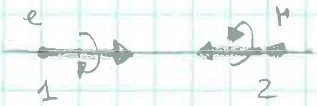
(adjoint / vertex / spinor)

$$(2\pi)^4 \int d^4q\; \delta^4(p_1-p_3-q)\;\delta^4(p_2+q-p_4)$$

$$\times \left[\bar{u}(4)\; i g_e \gamma^\mu u(2)\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{u}(3)\; i g_e \gamma^\nu u(1)\right]$$

$$\mathcal{M} = -\frac{g_e^2}{(p_1-p_3)^2}\left[\bar{u}(4)\gamma^\mu u(2)\right]\left[\bar{u}(3)\gamma_\mu u(1)\right]$$

Consider all particles have helicity +1 in initial and final state



Initial state

Final state

$p_x = p_z = 0$ if we assume particles move along z, hence

$p_z = |\vec{p}| = \sqrt{E^2 - m^2} = \sqrt{(E+m)(E-m)}$ for 1 and 4 but $p_z = -|\vec{p}|$ for 2 and 3

Under this assumption the Dirac spinors become

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{\sqrt{(E+m)(E-m)}}{(E+m)} \\ 0 \end{pmatrix} = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \sqrt{\frac{E-m}{E+m}} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix} \quad \text{where } N = \sqrt{E+m}$$

$$u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}$$

Using an abbreviation $a_\pm \equiv \sqrt{E_e \pm m_e}$ and $b_\pm \equiv \sqrt{E_\mu \pm m_\mu}$

we can write the spinors for our initial and final state electron and muon

So :

$$u(1) = \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix} \quad u(2) = \begin{pmatrix} 0 \\ b_+ \\ 0 \\ b_- \end{pmatrix} \quad u(3) = \begin{pmatrix} 0 \\ a_+ \\ 0 \\ a_- \end{pmatrix} \quad u(4) = \begin{pmatrix} b_+ \\ 0 \\ b_- \\ 0 \end{pmatrix}$$

N.B. We have b_- and not $-b_-$ because $P_z = -|\vec{p}|$ for $u(1)$

+ same here

$$\mathcal{M} = -\frac{g_e^2}{(P_1-P_3)^2}\left\{[\bar{u}(4)\gamma^0 u(2)][\bar{u}(3)\gamma_0 u(1)] - [\bar{u}(4)\gamma^i u(2)][\bar{u}(3)\gamma_i u(1)]\right\}$$

where i is summed from 1 to 3

$$-\bar{u}(3)\gamma_0 u(1) = (0\ a_+\ 0\ a_-)\gamma_0\gamma_0 \begin{pmatrix} 0 \\ a^+ \\ 0 \\ a_- \end{pmatrix} = 0 \quad \text{because } (\gamma_0)^2 = 1$$

same for $[\bar{u}(4)\gamma_0 u(2)]$

$$-\bar{u}(3)\gamma_i u(1) = (0 a_+\ 0\ a_-)\gamma_0\gamma_i \begin{pmatrix} 0 \\ a^+ \\ 0 \\ a_- \end{pmatrix} = (0\ a_+\ 0\ a_-)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} a^+ \\ 0 \\ a_- \\ 0 \end{pmatrix} =$$

$$= (0 a_+\ 0 - a_-)\begin{pmatrix} \sigma^i\begin{pmatrix} a_- \\ 0 \end{pmatrix} \\ -\sigma^i\begin{pmatrix} a^+ \\ 0 \end{pmatrix} \end{pmatrix} = (0 a_+)\sigma^i\begin{pmatrix} a_- \\ 0 \end{pmatrix} + (0 + a_-)(+\sigma^i)\begin{pmatrix} a^+ \\ 0 \end{pmatrix} =$$

$$= 2a_+a_-\,(0\ 1)\ \sigma^i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2a_+a_-\,(0\ 1)\begin{pmatrix} \sigma^i_{11} & \sigma^i_{12} \\ \sigma^i_{21} & \sigma^i_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= 2a_+a_- \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11}^i \\ \sigma_{21}^i \end{pmatrix} = 2a_+a_- \sigma_{21}^i$$

$$\bar{u}(4)\gamma^i u(2) = \begin{pmatrix} b_+ & 0 & b_- & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ b^+ \\ 0 \\ b_- \end{pmatrix} =$$

$$= \begin{pmatrix} b_+ & 0 & -b_- & 0 \end{pmatrix} \begin{pmatrix} \sigma^i \begin{pmatrix} 0 \\ b_- \end{pmatrix} \\ -\sigma^i \begin{pmatrix} 0 \\ b^+ \end{pmatrix} \end{pmatrix} = \begin{pmatrix} b_+ & 0 \end{pmatrix} \sigma^i \begin{pmatrix} 0 \\ b_- \end{pmatrix} + \begin{pmatrix} +b_- & 0 \end{pmatrix} \sigma^i \begin{pmatrix} 0 \\ b^+ \end{pmatrix} =$$

$$= 2b_+b_- \left[\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{11}^i & \sigma_{12}^i \\ \sigma_{21}^i & \sigma_{22}^i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = 2b_+b_- \left[\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{12}^i \\ \sigma_{22}^i \end{pmatrix} \right] = 2b_+b_- \sigma_{12}^i$$

$$M = \frac{g_e^2}{(P_1 - P_3)^2} (2a_+a_- 2b_+b_-) \sigma_{21} \cdot \sigma_{12} = \frac{8g_e^2}{(P_1 - P_3)^2} (a_+a_-)(b_+b_-)$$

$$\boxed{\begin{aligned} &\text{But } \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\text{hence } \sigma_{21} \quad \sigma_{12} = (1)(1) + (i)(-i) + (0)(0) = 2 \end{aligned}}$$

$$M = \frac{8g_e^2}{(P_1 - P_3)^2} (a_+a_-)(b_+b_-)$$

$$(a_+a_-) = \sqrt{E_e^2 - m_e^2} = \sqrt{|\vec{P}_e|^2} = |\vec{P}_e|$$

and $|\vec{P}_e| = |\vec{P}_p|$

$$(b_+b_-) = \sqrt{E_p^2 - m_p^2} = \sqrt{|\vec{P}_p|^2} = |\vec{P}_p|$$

But $P_1 = (E_e, \vec{P}_e)$, $P_3 = (E_e, -\vec{P}_e)$

So $M = \dfrac{8g_e^2}{(P_1 - P_3)^2} |\vec{P}_e|^2$

so $(P_1 - P_3) = (0, 2\vec{P}_e)$ and $(P_1 - P_3)^2 = 0 - 4|\vec{P}_e|^2$

$$\boxed{M = \frac{8g_e^2}{-4|\vec{P}_e|^2} |\vec{P}_e|^2 = -2g_e^2}$$

CHARGE CONJUGATION

$$\phi_c = i\gamma^2 \psi^*$$

Find the charge conjugation of Dirac equation solution u_1 and u_2

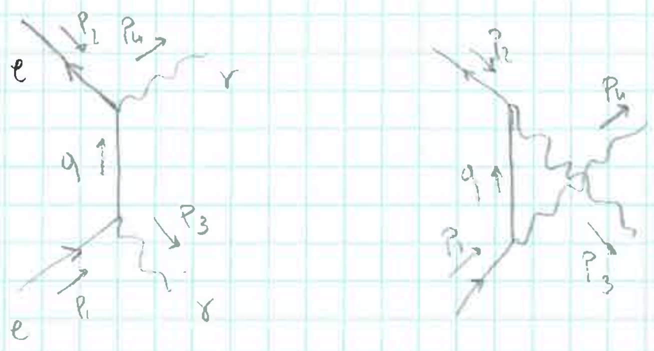
$$i\gamma^2 = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$i\gamma^2 u^{(1)*} = N\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{(p_x - ip_y)}{E+m} \end{pmatrix} = \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v^{(1)}$$

$$i\gamma^2 u^{(2)*} = N\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x + ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix} = N\begin{pmatrix} -\frac{p_z}{E+m} \\ -\frac{p_x + ip_y}{E+m} \\ -1 \\ 0 \end{pmatrix} = v^{(2)}$$

FEYNMAN RULES

a) $e^+ + e^- \longrightarrow \gamma + \gamma$



1^{st} diagram

$$(2\pi)^4 \int d^4q \, \delta^4(p_1 - q - p_3)\, \delta^4(p_2 + q - p_4)$$

$$\times\, \varepsilon^*_\mu(4) \left[\overline{v}(2)\, i g_e \gamma^\mu \frac{i(\not{q} + m)}{q^2 - m^2}\, i g_e \gamma^\nu u(1) \right] \varepsilon^*_\nu(3) =$$

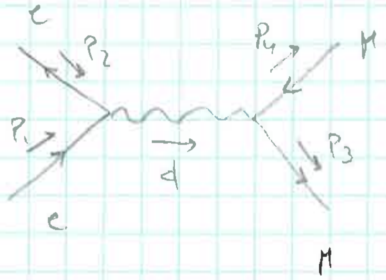
$$= (2\pi)^4 (-i g_e^2) \left[\overline{v}(2)\, \not{\varepsilon}^*(4) \frac{(\not{p}_1 - \not{p}_3 + m)}{(p_1 - p_3)^2 - m^2}\, \not{\varepsilon}^*(3)\, u(1) \right]$$

$$\mathcal{M}_1 = \frac{g_e^2}{(p_1 - p_3)^2 - m^2} \left[\overline{v}(2)\, \not{\varepsilon}^*(4)\, (\not{p}_1 - \not{p}_3 + m)\, \not{\varepsilon}^*(3)\, u(1) \right]$$

2^{nd} diagram is identical except for $3 \leftrightarrow 4$ swapping

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1 - p_4)^2 - m^2} \left[\overline{v}(2)\, \not{\varepsilon}^*(3)\, (\not{p}_1 - \not{p}_4 + m)\, \not{\varepsilon}^*(4)\, u(1) \right]$$

b) $e^+ + e^- \longrightarrow \mu^+ + \mu^-$

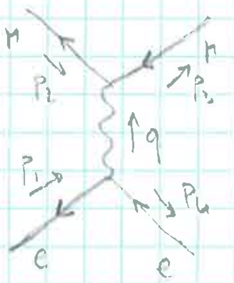


$$(2\pi)^4 \int d^4q \, \delta^4(p_1 + p_2 - q) \, \delta^4(q - p_4 - p_3) \times$$

$$\times \left[\bar{v}(2) \, i g_e \gamma^\mu u(1) \right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{u}(3) \, i g_e \gamma^\nu v(4) \right]$$

$$\mathcal{M} = \frac{g_e^2}{(p_1 + p_2)^2} \left[\bar{v}(2) \gamma^\mu u(1) \right] \left[\bar{u}(3) \gamma_\mu v(4) \right]$$

c) $e^+ + \mu^+ \longrightarrow e^+ + \mu^+$



$$(2\pi)^2 \int d^4q \, \delta^4(p_1 - p_4 - q) \, \delta^4(p_2 + q - p_3) \times$$

$$\times \left[\bar{v}(2) \, i g_e \gamma^\mu v(3) \right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{v}(1) \, i g_e \gamma^\nu v(4) \right]$$

$$\mathcal{M} = \frac{g_e^2}{(p_1 - p_4)^2} \left[\bar{v}(2) \gamma^\mu v(3) \right] \left[\bar{v}(1) \gamma_\mu v(4) \right]$$