Exercise 1. *Instantons*

In four-dimensional Euclidean space ($\mu, \nu = 1, 2, 3, 4$), consider the instanton configuration of a $SU(2)$ gauge field

$$\underline{A}^a_\mu = \frac{2}{g} \frac{\eta^a_{\mu\nu} x_\nu}{|x|^2 + \lambda^2}\,, \tag{1}$$

where λ is an arbitrary distance and η is 't Hooft's mixed colour and space-time tensor defined as

$$\eta^a_{\mu\nu} = \epsilon_{a\mu\nu 4} - \delta_{\mu 4}\delta_{a\nu} + \delta_{\nu 4}\delta_{a\mu} \tag{2}$$

or, equivalently,

$$\eta^a_{\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \text{if } \mu \neq 4,\ \nu \neq 4 \\ \delta_{a\mu} & \text{if } \mu \neq 4,\ \nu = 4 \\ -\delta_{a\nu} & \text{if } \mu = 4,\ \nu \neq 4 \\ 0 & \text{if } \mu = \nu = 4\,. \end{cases} \tag{3}$$

The tensor η satisfies the contraction identities:

$$\begin{aligned} \eta^a_{\mu\nu} &= -\eta^a_{\nu\mu} \\ \epsilon_{abc}\eta^b_{\mu\nu}\eta^c_{\rho\sigma} &= \delta_{\mu\rho}\eta^a_{\nu\sigma} - \delta_{\mu\sigma}\eta^a_{\nu\rho} - \delta_{\nu\rho}\eta^a_{\mu\sigma} + \delta_{\nu\sigma}\eta^a_{\mu\rho} \\ \eta^a_{\mu\nu}\eta^a_{\mu\nu} &= 12\,. \end{aligned} \tag{4}$$

1. Show that the field strength tensor $\underline{F}^a_{\mu\nu} = \partial_\mu \underline{A}^a_\nu - \partial_\nu \underline{A}^a_\mu + g\epsilon_{abc}\underline{A}^b_\mu \underline{A}^c_\nu$ is given by

$$\underline{F}^a_{\mu\nu} = \frac{4}{g} \frac{-\lambda^2 \eta^a_{\mu\nu}}{\left(|x|^2 + \lambda^2\right)^2}\,. \tag{5}$$

2. Compute the action corresponding to this field configuration,

$$\underline{S} = \frac{1}{4} \int \mathrm{d}^4 x \underline{F}^a_{\mu\nu} \underline{F}^a_{\mu\nu}\,. \tag{6}$$