MMP 2
Solutions 9

FS 19
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Exercise 1

$$[G,G] := \left\{ aba^{-1}b^{-1} \quad | \; a,b \in G \right\}$$

a) $[G,G]$ normal subgroup of $G \quad \Longleftrightarrow \quad c[G,G]c^{-1} \subseteq [G,G] \quad \forall c \in G$

$$\Rightarrow \quad c[G,G]c^{-1} = caba^{-1}b^{-1}c^{-1} \underset{\substack{\uparrow \\ \text{insert neutral element}}}{=} cac^{-1}cbc^{-1}ca^{-1}c^{-1}cb^{-1}c^{-1}$$

$$= \underbrace{(cac^{-1})}_{:=A}\underbrace{(cbc^{-1})}_{:=B}\underbrace{(cac^{-1})^{-1}}_{A^{-1}}\underbrace{(cbc^{-1})^{-1}}_{B^{-1}}$$

$$= \underbrace{ABA^{-1}B^{-1}}_{\in [G,G]!} \qquad \text{with } A,B \in G$$

$\square$

b)

$[g] :=$ class of g in $G/_{[G,G]}$
with the product: $[g_1][g_2] = [g_1 g_2]$
factor group: $G/_{[G,G]}$ with the product above
equivalence relation: $[g_1] = [g_2] \quad \Leftrightarrow \quad \exists h \in [G,G]$ with $g_2 = g_1 h$

$$G/_{[G,G]} \text{ abelian} \quad \Leftrightarrow [g_1][g_2] = [g_2][g_1]$$
$$\downarrow$$
$$[g_1 g_2] = [g_2 g_1]$$
$$\downarrow$$
$$g_1 g_2 \sim g_2 g_1$$
$$\downarrow$$
$$\exists h \in [G,G] \textbf{ such that } g_1 g_2 = h g_2 g_1$$
$$\Rightarrow \quad h = g_1 g_2 g_1^{-1} g_2^{-1} \in [G,G]$$

$\square$

Exercise 2

$\mu : G \times X \to X$
$\mu(e, x) = x \quad \forall x \in X$
$\mu(h, \mu(g, x)) = \mu(hg, x)$

$G_x := \{g \in G \ \mid \mu(g, x) = x\}$ is the <u>stabilizer</u> of G with respect to $x \in X$

To show: G_x is a subgroup of G.

1) <u>neutral Element</u>: $\mu(e, x) = x \quad \forall x \in X \quad \Rightarrow \quad e \in G_x$

2) <u>inverse element</u>: Let $g \in G_x$, then:

$$\mu(g^{-1}, x) \overset{g \in G_x}{=} \mu(g^{-1}, \mu(g, x)) = \mu(g^{-1}g, x) = \mu(e, x) = x \quad \Rightarrow \quad g^{-1} \in G_x$$

3) <u>closure</u>: $g_1, g_2 \in G_x \quad \Rightarrow \quad x = \mu(g_1, x) = \mu(g_1, \mu(g_2, x)) = \mu(g_1 g_2, x) \quad \Rightarrow \quad g_1 g_2 \in G_x \quad \square$

Exercise 3

a) ρ representation fo $G \quad \Leftrightarrow \quad \rho(g_1)\rho(g_2) = \rho(g_1 \circ g_2)\ , \quad g_1, g_2 \in G$

<u>ρ_1 :</u>

$$\underline{\rho_1(x_1) \cdot \rho_1(x_2)} = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + x_2 \\ 0 & 1 \end{pmatrix}$$
$$= \rho(x_1 + x_2)$$

$$\Rightarrow \quad \underline{G_1 = (\mathbb{R}, +)}$$

<u>ρ_2 :</u>

$$\underline{\rho_2(x_1) \cdot \rho_2(x_2)} = \begin{pmatrix} 1 & \log(x_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \log(x_2) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \log(x_1) + \log(x_2) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \log(x_1 x_2) \\ 0 & 1 \end{pmatrix} = \underline{\rho_2(x_1 \cdot x_2)}$$
$$\Rightarrow \underline{G_2 = (\mathbb{R}_+, \cdot)}$$

b)

$$\rho(x)\vec{v} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + xv_2 \\ v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \rho_1(x) \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad \forall a \in \mathbb{R}$$

$$\Rightarrow \quad V = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix}, \ a \in \mathbb{R} \right\} \text{ is an invariant subspasce under the action of } \rho_1.$$

c) $\rho(x)$ completely reducible

$$\Leftrightarrow \quad \exists \mu : \qquad \rho'(x) = \mu^{-1}\rho(x)\mu \quad \forall x$$

$$\text{such that} \quad \rho'(x) = \begin{pmatrix} \boxed{\rho^{(1)}} & & 0 \\ & \boxed{\ddots} & \\ 0 & & \boxed{\rho^{(n)}} \end{pmatrix} \qquad \text{with } n \text{ invariant subspaces}$$

two dimensional case:

$$\exists \mu : \rho'(x) = \mu^{-1}\rho(x)\mu = \begin{pmatrix} \rho^{(1)}(x) & 0 \\ 0 & \rho^{(2)}(x) \end{pmatrix} \qquad \forall x$$

$$\mu := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mu^{-1} = \frac{1}{\det \mu} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow \frac{1}{\det \mu} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\det \mu} \begin{pmatrix} \det \mu + cdx & d^2x \\ -c^2 & \det \mu - cdx \end{pmatrix}$$

$$\stackrel{!}{=} \begin{pmatrix} \rho^{(1)} & 0 \\ 0 & \rho^{(2)} \end{pmatrix}$$

$$\Rightarrow \quad \rho^{(1)}(x) = 1 + \frac{cd}{\det \mu}(x)$$

$$\rho^{(2)}(x) = 1 - \frac{cd}{\det \mu}(x)$$

$$\left.\begin{aligned} 0 &= d^2 x \\ 0 &= -c^2 x \end{aligned}\right\} \forall x \qquad \Rightarrow \quad c = d = 0$$

$$\Rightarrow \quad \mu = \begin{pmatrix} q & b \\ 0 & 0 \end{pmatrix} \qquad \det \mu = 0$$

$$\Rightarrow \quad \nexists \mu^{-1} \quad \Rightarrow \nexists \mu!$$

$\Rightarrow$ <u>is not completely reducible!</u>

<u>Alternative ways:</u>

ρ_1 completely reducible $\quad \Leftrightarrow \quad \exists$ invariant subspaces V, V' such that

$$\underbrace{\mathbb{R}^2}_{\text{where } \rho_1 \text{ acts upon}} = V \oplus V'$$

1) V invariant subspace $\quad \Leftrightarrow \quad \rho_1(x)V = V \quad \Leftrightarrow \quad \rho_1(x)v = \lambda v \quad \forall v \in V$

$\Rightarrow \quad \det(\rho_1(x) - \lambda) = 0 = (\lambda - 1)^2 \quad \Rightarrow \quad \underset{\text{degenerated!}}{\lambda_{1,2} = 1}$

corresponding eigenspace:

$$(\rho_1(x) - \lambda) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} xv_2 = 0 \quad & \forall x \quad \Rightarrow v_2 = 0 \\ 0 = 0 \quad & \forall v_1 \quad \Rightarrow v_1 \in \mathbb{R} \end{aligned}\right\} V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

These are all the eigenspaces, i.e. all the invariant subspaces.

$\Rightarrow \quad \nexists V'$ such that $\mathbb{R}^2 = V \oplus V' \quad \Rightarrow \quad$ <u>ρ_1 not completely reducible</u>

2) ρ_1 completely reducible $\quad \Leftrightarrow \quad \rho_1$ can be diagonalized (only in 2d)
$\Leftrightarrow \quad \exists \mu$ s.t. $\rho_1' = \mu^{-1}\rho_1\mu$ with $\mu = (v_1, v_2)$, v_1, v_2 eigenvectors of $\rho_1 \quad \rightarrow \quad$ such two eigenvectors don't exist (as shown in 1)) $\Rightarrow \nexists \mu$
$\Rightarrow \quad \rho_1$ is not completely reducible.

Exercise 4

a)

$$\rho_1(x_1)\rho_1(x_2) = \begin{pmatrix} 1 & x_1 & x_1 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 & x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + x_2 & x_1 + x_2 + x_1 x_2 \\ 0 & 1 & x_1 + x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\neq \rho_1(x_1 + x_2) = \begin{pmatrix} 1 & x_1 + x_2 & x_1 + x_2 \\ 0 & 1 & x_1 + x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\rho_1 \text{ is not a representation of } (\mathbb{R}, +)}$$

b)

$$\rho_2(x_1)\rho_2(x_2) = \begin{pmatrix} 1 & x_1 & \frac{x_1^2}{2} \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 & \frac{x_2^2}{2} \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + x_2 & \frac{1}{2}(x_1 + x_2)^2 \\ 0 & 1 & x_1 + x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \underline{\rho_2(x_1 + x_2)}$$

$$\Rightarrow \quad \underline{\rho_2 \text{ is a representation of } (\mathbb{R}, +)}$$

c)

$$\rho_3(e^{i\varphi_1})\rho_3(e^{i\varphi_2}) = \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2 \\ \sin\varphi_1 \cos\varphi_2 + \cos\varphi_1 \sin\varphi_2 \end{pmatrix}}_{\sin(\varphi_1+\varphi_2)} \underbrace{\begin{pmatrix} -\cos\varphi_1 \sin\varphi_2 - \sin\varphi_1 \cos\varphi_2 \\ -\sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \end{pmatrix}}_{\cos(\varphi_1+\varphi_2)}$$

$$= \begin{pmatrix} \cos(\varphi_1 + \varphi_2) & -\sin(\varphi_1 + \varphi_2) \\ \sin(\varphi_1 + \varphi_2) & \cos(\varphi_1 + \varphi_2) \end{pmatrix}$$

$$= \rho_3(e^{i\varphi_1} \cdot e^{i\varphi_2}) = \rho_3(e^{i(\varphi_1+\varphi_2)})$$

$$\underline{\Rightarrow \quad \rho_3 \textbf{ is a prepresentation of } U(1)}$$