Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 4

Exercise 1: Dirac spinors

a) Construct spinor eigenstates $u(p,\sigma)$, $v(p,\sigma)$, $\sigma = \pm$ in the z-direction.

b) Verify the validity of the spinor completeness relations:

$$\sum_{\sigma=+,-} u(p,\sigma)\bar{u}(p,\sigma) \;=\; \not{p} \;+\; m\mathbf{1}\,, \tag{1}$$

$$\sum_{\sigma=+,-} v(p,\sigma)\bar{v}(p,\sigma) \;=\; \not{p} \;-\; m\mathbf{1}\,, \tag{2}$$

where $\mathbf{1}$ is the unit 4×4 matrix.

Exercise 2: Fermion and chiralities

We will consider fermions described by the Lagrangian

$$\mathcal{L} \;=\; \bar{\psi}(i\not{\partial} \;-\; m)\psi\,. \tag{3}$$

a) Without using any particular representation for the Dirac gamma matrices, show that the following relations involving the chiral operator $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ hold:

$$\begin{aligned} (\mathbb{P}_{R,L})^2 &= \mathbb{P}_{R,L}\,, \\ \mathbb{P}_R\mathbb{P}_L &= \mathbb{P}_L\mathbb{P}_R \;=\; 0\,, \\ \gamma^5\,\psi_{R,L} &= \pm\psi_{R,L}\,, \end{aligned}$$

where $\mathbb{P}_{R,L} := (1 \pm \gamma^5)/2$ and $\psi_{R,L} := \mathbb{P}_{R,L}\,\psi$. Notice that the first two relations above imply that $\mathbb{P}_{R,L}$ are projector operators and the last relation that the $\psi_{R,L}$ fields are eigenstates of γ^5 with definite chiralities.

b) For a massless fermion, show that the Lagrangian (3) is invariant under the chiral transformations $\psi \to U\psi$ defined by

$$U \;:=\; \exp\left(-i\alpha\gamma_5\right). \tag{4}$$

Show that the mass term $(m \neq 0)$ breaks this symmetry.

c) Re-write the Dirac Lagrangian using the chiral fields $\psi_{L,R}$ and derive the equations of motions for $\psi_{L,R}$. Find the associated conserved current when $m = 0$.

Exercise 3: Forward-backward asymmetry at electron-positron colliders

Consider the interacting Lagrangian for electrons ψ_e and muons ψ_μ coupled to a massless vector field A_ν

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}} \tag{5}$$

with

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{6}$$

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}_e(i\partial\!\!\!/ - m_e)\psi_e + \bar{\psi}_\mu(i\partial\!\!\!/ - m_\mu)\psi_\mu \tag{7}$$

$$\mathcal{L}_{\text{int}} = g_V\, A_\nu(\bar{\psi}_e\gamma^\nu\psi_e + \bar{\psi}_\mu\gamma^\nu\psi_\mu) + g_A\, A_\nu(\bar{\psi}_e\gamma^\nu\gamma^5\psi_e + \bar{\psi}_\mu\gamma^\nu\gamma^5\psi_\mu)\,, \tag{8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and g_V and g_A are vector and axial-vector coupling constants, respectively. In this problem we will study the scattering process $e^-e^+ \to \mu^-\mu^+$ for this model at a high-energy electron-positron collider at fixed collision energy.

a) Is Eq. (5) invariant under the chiral rotation defined in Eq. (4)? Which discrete symmetries are violated for non-zero g_A?

b) Write down the Feynman rules for this model. Draw the tree-level Feynman diagrams for the collision processes $e^-e^+ \to e^-e^+$ and $e^-e^+ \to \mu^-\mu^+$.

c) Derive the scattering amplitude $i\mathcal{M}$ for $e^-(p_1)\, e^+(p_2) \to \mu^-(p_3)\, \mu^+(p_4)$ using Feynman's gauge ($\xi = 1$) in the massless limit $m_e = m_\mu = 0$.

d) Compute that the spin-averaged squared amplitude $\overline{|M|^2} := \frac{1}{4}\sum_{\sigma=+,-}|\mathcal{M}|^2$. You should find:

$$\overline{|M|^2} = \frac{1}{4(p_1+p_2)^4}\,\text{Tr}\left[p\!\!\!/_1(g_V\mathbf{1} - g_A\gamma^5)\gamma^\mu p\!\!\!/_2\gamma^\nu(g_V\mathbf{1} + g_A\gamma^5)\right] \times \text{Tr}\left[p\!\!\!/_4(g_V\mathbf{1} - g_A\gamma^5)\gamma_\mu p\!\!\!/_3\gamma_\nu(g_V\mathbf{1} + g_A\gamma^5)\right]\,.$$

e) Work out the traces and kinematics to show that:

$$\overline{|M|^2} = 2\left(g_V^2 - g_A^2\right)^2\frac{u^2}{s^2} + 2\left(g_V^4 + g_A^4 + 6\,g_V^2 g_A^2\right)\frac{t^2}{s^2}$$

where s, t and u are the Mandelstam variables, introduced in problem sheet 1 (Ex. 4).

Hint: You may use the following useful identities $\text{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = 4(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha})$ and $\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\sigma\rho} = 2(\delta^\mu_\sigma\delta^\nu_\rho - \delta^\mu_\rho\delta^\nu_\sigma)$.

f) Compute the total scattering cross-section of $e^-e^+ \to \mu^-\mu^+$ at a fixed center-of-mass collision energy $E_{CM} = \sqrt{s}$. The results is:

$$\sigma_{\text{tot}} = \frac{(g_V^2 + g_A^2)^2}{12\pi\, E_{CM}^2}\,.$$

Hint: First show that the differential cross-section $\frac{d\sigma}{d\Omega}$ can be brought to the form $\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\overline{|M|^2}$, then integrate over t with boundaries $t \in [-s, 0]$.

An important observable widely used at high-energy colliders like the Large Electron-Positron collider (LEP) at CERN and later at the LHC is the *forward-backward asymmetry*

$$\mathcal{A}_{FB} \;=\; \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}\,. \tag{9}$$

The forward and backward cross-sections, σ_F and σ_B, are defined by integrating the differential cross-section over angles confined to each spherical hemisphere

$$\sigma_F \;:=\; \int_0^1 \mathrm{d}(\cos\theta)\,\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\,, \qquad \sigma_B \;:=\; \int_{-1}^0 \mathrm{d}(\cos\theta)\,\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\,, \tag{10}$$

where θ is the angle between the directions of the produced muon μ^- and the colliding electron beam.

g) Show that the differential cross-section for the collision process $e^-e^+ \to \mu^-\mu^+$ in the center-of-mass frame can be brought to the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} \;=\; A(s) \;+\; B(s)\cos\theta \;+\; C(s)\cos^2\theta$$

where A, B and C are functions of s, to be determined explicitly.

h) Compute the forward-backward asymmetry $\mathcal{A}_{FB}$. For which values of the couplings $g_{V,A}$ is this observable maximized? What would be the value of $\mathcal{A}_{FB}$ predicted by QED?

i) Re-write (5) using the chiral fields $\psi_{\ell_L} := \mathbb{P}_L\,\psi_\ell$ and $\psi_{\ell_R} := \mathbb{P}_R\,\psi_\ell$ for $\ell = e, \mu$. What is the value of the forward-backward asymmetry if the vector-boson A_ν interacts exclusively with left-handed electrons ψ_{e_L} and left-handed muons ψ_{μ_L}? Argue that $\mathcal{A}_{FB}$ is a probe of parity violation.