Elementary Particle Theory – PHY452

Fall Semester 2020

Exercise Sheet 5

Exercise 1: Gauge invariance

Consider a multiplet of quantum fields $\Psi = (\psi_1, \ldots \psi_n)$ transforming according to a unitary representation of a compact gauge group G

$$\Psi(x) \to \Psi'(x) = U(x)\Psi(x), \qquad U(x) = \exp\left(i\hat{\theta}(x)\right) = \exp\left(i\theta^a(x)t^a\right) . \tag{1}$$

To construct a gauge invariant Lagrangian we must define a covariant derivative that satisfies the transformation property

$$D_\mu\Psi(x) \to D'_\mu\Psi'(x) = U(x)D_\mu\Psi(x) . \tag{2}$$

To this end we introduce a *parallel transport* operator $V(x, y)$ with the transformation behaviour

$$V(y, x) \to U(y)V(y, x)U^{-1}(x) , \tag{3}$$

and define the covariant derivative as

$$n^\mu D_\mu \Psi = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\Psi(x + \epsilon n) - V(x + \epsilon n, x)\, \Psi(x)\right] . \tag{4}$$

The operator V can be written in terms of a *connection* $\hat{A}_\mu(x) = A^a_\mu(x)\, t^a$ such that

$$V(x + \epsilon n, x) = 1 + ig\epsilon n^\mu \hat{A}_\mu + \mathcal{O}(\epsilon^2) . \tag{5}$$

Find the explicit expression for the covariant derivative.

Exercise 2: Massive vector boson

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu , \tag{6}$$

describing a massive gauge boson A_μ with mass $m \neq 0$ and with field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

a) Derive the explicit form of the Euler-Lagrange equations,

$$\partial_\alpha \frac{\delta\mathcal{L}}{\delta(\partial_\alpha A_\beta)} - \frac{\delta\mathcal{L}}{\delta A_\beta} = 0 . \tag{7}$$

Hint: *Replace all contravariant indices by covariant ones by using the metric tensor, since the derivative in Eq. (7) are taken w.r.t. covariant objects, e.g.* $X^\mu = g^{\mu\nu}X_\nu$.

b) Show that the equations of motion imply the condition $\partial_\mu A^\mu = 0$, which reduces them to a set of Klein-Gordon equations.

c) Discuss the behaviour of the Lagrangian and the equations of motion with respect to gauge transformations $A_\mu \to A_\mu + \partial_\mu \chi$ which leave $F_{\mu\nu}$ invariant.

d) The equation for the Green's function in spacetime coordinates reads

$$\left[\left(\Box + m^2\right) g^{\mu\nu} - \partial^\mu \partial^\nu\right] D_{\nu\rho}(x) = i\, g^\mu_\rho \delta^{(4)}(x)\,, \tag{8}$$

which can be expressed in momentum space as follows,

$$\left(-k^2 g^{\mu\nu} + k^\mu k^\nu + m^2 g^{\mu\nu}\right) D_{\nu\rho}(k) = i\, g^\mu_\rho\,. \tag{9}$$

Show that the Proca propagator $D_{\nu\rho}(k)$ is given by

$$D^{\nu\rho}(k) = \frac{-i}{k^2 - m^2 + i\epsilon}\left(g^{\nu\rho} - \frac{k^\nu k^\rho}{m^2}\right)\,. \tag{10}$$

Hint: *Decompose the l.h.s of Eq. (9) in terms of the longitudinal and transverse projectors:*

$$P_L^{\mu\nu}(k) = \frac{k^\mu k^\nu}{k^2}\,, \qquad P_T^{\mu\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\,. \tag{11}$$

Exercise 3: How many neutrinos species?

Neutrinos (ν's) are neutral and massless fermions in the Standard Model (SM) that interact with others particles only via the electroweak interactions, which are mediated by the $W^\pm$ and Z vector bosons. The interaction between ν's and the Z-boson is described by the following Lagrangian,

$$\mathcal{L}_Z = \frac{m_Z}{v}\sum_{i=1}^{N} \bar{\nu}_i \gamma_\mu \omega_L \nu_i\, Z^\mu\,, \tag{12}$$

where $\omega_L = (1-\gamma^5)/2$, $m_Z \approx 91$ GeV is the Z-boson mass, $v = (\sqrt{2}G_F)^{-1/2} \approx 246$ GeV and $i = \{1, 2, ..., N\}$ are indices for the number of neutrino species. [1] The goal of this exercise is to determine the number of neutrinos N from measurements performed at the LEP experiment.

a) Write down the Feynman rules from Eq. (12) and compute the $Z \to \nu_i \bar{\nu}_i$ amplitude.

b) Show that the averaged squared matrix-element for this process is given by

$$\overline{\sum} |\mathcal{M}(Z \to \nu_i \bar{\nu}_i)|^2 = \frac{4(k_1 \cdot k_2)}{3}\frac{m_Z^2}{v^2}\,, \tag{13}$$

where k_1 and k_2 denote the neutrino and anti-neutrino momentum, respectively.

Reminder: *The Dirac matrices satisfy*

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho) = 4(g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\sigma})\,. \tag{14}$$

The sum over the polarizations vectors $\varepsilon_\mu(k,\sigma)$ of a massive vector boson V with momentum k^μ is given by

$$\sum_\lambda \varepsilon_\mu(k,\lambda)\, \varepsilon_\nu(k,\lambda)^* = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_V^2}\,, \tag{15}$$

where m_V denotes its mass.

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[1] $G_F \approx 1.1 \times 10^{-5}$ GeV^{-2} is the Fermi coupling constant which will be discussed in the following lectures.

c) Show that the total width of the Z-boson into neutrinos is given by,

$$\Gamma_{\text{inv}} = \sum_{i=1}^{N} \Gamma(Z \to \nu_i \bar{\nu}_i) = \frac{N\, m_Z^3}{24\pi v^2}\,. \tag{16}$$

d) Γ_{inv} is called the total Z-boson invisible width since neutrinos are always undetected at collider experiments. By measuring the total Z-boson width and subtracting all the relevant visible decay channels, LEP was able to determine

$$\Gamma_{\text{inv}}^{\text{exp}} = 499.0(1.5)\ \text{MeV}\,. \tag{17}$$

By comparing Eq. (16) and (17), estimate the number of neutrinos species N in the SM.