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General Relativity
Sheet 6

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Prof. P. Jetzer

M. Haney, S. Tiwari, A. Boitier, M. Ebersold, A. Dei, L. Fritz
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Exercise 1 [Tensors in General Relativity]

Consider a general coordinate transformation $x'^\mu = x'^\mu(x^\alpha)$. Let $V^\mu(x^\alpha)$ be a vector field, and ${x_{(i)}}^\mu(\tau)$ the worldline of a particle, parameterized by its proper time τ. How do the following quantities change under this transformation?

a) $g_{\mu\nu}$

b) ${\Gamma^\mu}_{\nu\rho}$

c) $A^\mu = \dfrac{d{x_{(1)}}^\mu}{d\tau}$

d) $B^\mu = \dfrac{d^2{x_{(1)}}^\mu}{d\tau^2}$

e) $\epsilon^{\mu\nu\rho\sigma}$

f) $|\det(g_{\mu\nu})|^{-1/2}\,\epsilon^{\mu\nu\rho\sigma}$

g) ${C^\mu}_\nu = \dfrac{\partial V^\mu}{\partial x^\nu}$

h) ${D^\mu}_\nu = \dfrac{\partial V^\mu}{\partial x^\nu} + {\Gamma^\mu}_{\nu\rho}V^\rho$

Are these tensors? Pseudotensors? Compare to the results of exercies 1 on sheet 3.

Exercise 2 [Equations of motion and coordinate frames]

Consider the metric given by

$$ds^2 = \left[1 - \omega^2\left(x^2 + y^2\right)\right]dt^2 + 2\omega y dt dx - 2\omega x dt dy - dx^2 - dy^2 - dz^2,$$

with the inverse metric given in matrix form as

$$g^{\mu\nu} = \begin{pmatrix} 1 & \omega y & -\omega x & 0 \\ \omega y & -\left(1-\omega^2 y^2\right) & -\omega^2 xy & 0 \\ -\omega x & -\omega^2 xy & -\left(1-\omega^2 x^2\right) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

i) Compute the Christoffel symbols, and show that the geodesic equation for a free particle in this metric is given by

$$\frac{d^2t}{d\tau^2} = 0 \qquad \frac{d^2x}{d\tau^2} = \omega\frac{dt}{d\tau}\left(\omega x\frac{dt}{d\tau} + 2\frac{dy}{d\tau}\right)$$

$$\frac{d^2z}{d\tau^2} = 0 \qquad \frac{d^2y}{d\tau^2} = \omega\frac{dt}{d\tau}\left(\omega y\frac{dt}{d\tau} - 2\frac{dx}{d\tau}\right)$$

ii) Express the equations of motion in a rotating coordinate frame defined by

$$\begin{aligned} t' &= t \\ x' &= x\cos\omega t - y\sin\omega t \\ y' &= x\sin\omega t + y\cos\omega t \\ z' &= z \end{aligned}$$

What is the metric in this coordinate system?

Exercise 3 [The sphere]

i) Show that the 2-sphere, i.e. the surface

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$$

is a differentiable manifold. To this end construct charts $\psi_i^\pm$ on $\{\pm x_i > 0\}$, and show that the transition functions are smooth (*Hint: it is sufficient to show this for just one transition function*).

ii) For this part use only the chart $\psi_1^+ : (x_1, x_2, x_3) \to (u, v)$. Find the components of the two basis vectors

$$X_u = \partial_u = a^\mu \frac{\partial}{\partial x^\mu}, \qquad X_v = \partial_v = b^\mu \frac{\partial}{\partial x^\mu}, \qquad \mu \in \{1, 2, 3\}$$

with respect to the partial derivatives of $\mathbb{R}^3$ by calculating $X_u(f|_{S^2})$ and $X_v(f|_{S^2})$ at a points $p \in S^2$, where f is a differentiable function on $\mathbb{R}^3$, i.e. calculate

$$X_u(f|_{S^2}) = \partial_u(f \circ (\psi_1^+)^{-1}), \qquad X_v(f|_{S^2}) = \partial_v(f \circ (\psi_1^+)^{-1})$$

iii) Find the integral curves of the two basis vector fields by solving the equations

$$\dot{\gamma}_u(t) = X_u(\gamma(t)), \qquad \dot{\gamma}_v(t) = X_v(\gamma_v(t)),$$

for the differentiable curves

$$\gamma_u(t) = (\gamma_{u1}(t), \gamma_{u2}(t), \gamma_{u3}(t)), \qquad \gamma_v(t) = (\gamma_{v1}(t), \gamma_{v2}(t), \gamma_{v3}(t)).$$